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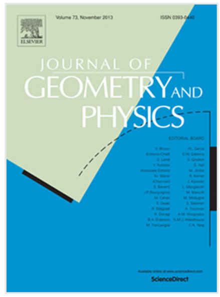
Citation for published version:
Pedrini, M, Sala, F & Szabo, RJ 2016, 'AGT relations for abelian quiver gauge theories on ALE spaces', *Journal of Geometry and Physics*, vol. 103, pp. 43–89. https://doi.org/10.1016/j.geomphys.2016.01.004

Digital Object Identifier (DOI):
10.1016/j.geomphys.2016.01.004

Published In:
Journal of Geometry and Physics

Accepted Manuscript



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PII: S0393-0440(16)00005-X
DOI: http://dx.doi.org/10.1016/j.geomphys.2016.01.004
Reference: GEOPHY 2678

To appear in: *Journal of Geometry and Physics*

Received date: 23 June 2014
Revised date: 3 November 2015
Accepted date: 14 January 2016

Please cite this article as: M. Pedrini, F. Sala, R.J. Szabo, AGT relations for abelian quiver gauge theories on ALE spaces, *Journal of Geometry and Physics* (2016), http://dx.doi.org/10.1016/j.geomphys.2016.01.004

EMPG–14–10

AGT RELATIONS FOR ABELIAN QUIVER GAUGE THEORIES ON ALE SPACES

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ABSTRACT. We construct level one dominant representations of the affine Kac-Moody algebra $\widehat{\mathfrak{gl}}_k$ on the equivariant cohomology groups of moduli spaces of rank one framed sheaves on the orbifold compactification of the minimal resolution X_k of the A_{k-1} toric singularity $\mathbb{C}^2/\mathbb{Z}_k$. We show that the direct sum of the fundamental classes of these moduli spaces is a Whittaker vector for $\widehat{\mathfrak{gl}}_k$, which proves the AGT correspondence for pure $\mathcal{N}=2$ $U(1)$ gauge theory on X_k. We consider Carlsson-Okounkov type Ext-bundles over products of the moduli spaces and use their Euler classes to define vertex operators. Under the decomposition $\widehat{\mathfrak{gl}}_k \simeq \mathfrak{h} \oplus \widehat{\mathfrak{sl}}_k$, these vertex operators decompose as products of bosonic exponentials associated to the Heisenberg algebra $\mathfrak{h}$ and primary fields of $\widehat{\mathfrak{sl}}_k$. We use these operators to prove the AGT correspondence for $\mathcal{N}=2$ superconformal abelian quiver gauge theories on X_k.

Date: November 2015

2010 Mathematics Subject Classification: 14D20, 14D21, 14J80, 81T13, 81T60

Keywords: framed sheaves, ALE spaces, Kac-Moody algebras, vertex operators, supersymmetric gauge theories, AGT relations, conformal field theories

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1 Introduction and summary

1.1 AGT relations and ALE spaces

In this paper we study a new occurrence of the deep relations between the moduli theory of sheaves and the representation theory of affine/vertex algebras.

We are particularly interested in the kind of relations which come from gauge theory considerations. An important example of these relations is the AGT correspondence for gauge theories on $\mathbb{R}^4$: in [3] Alday, Gaiotto and Tachikawa conjectured a relation between the instanton partition functions of $\mathcal{N} = 2$ supersymmetric quiver gauge theories on $\mathbb{R}^4$ and the conformal blocks of two-dimensional A_{r-1} Toda conformal field theories (see also [62, 4]); this conjecture has been explicitly confirmed in some special cases, see e.g. [41, 2, 59, 1]. From a mathematical perspective, this correspondence implies: (1) the existence of a representation of the W-algebra $\mathcal{W}(\mathfrak{gl}_r)$ on the equivariant cohomology of the moduli spaces $\mathcal{M}(r, n)$ of framed sheaves on the projective plane $\mathbb{P}^2$ of rank r and second Chern class n such that the latter is isomorphic to a Verma module of $\mathcal{W}(\mathfrak{gl}_r)$; (2) the fundamental classes of $\mathcal{M}(r, n)$ give a Whittaker vector of $\mathcal{W}(\mathfrak{gl}_r)$ (pure gauge theory); (3) the Ext vertex operator is related to a certain "intertwiner" of $\mathcal{W}(\mathfrak{gl}_r)$ under the isomorphism stated in (1) (quiver gauge theory). The instances (1) and (2) were proved by Schiffmann and Vasserot [56], and independently by Maulik and Okounkov [40]. For $r = 1$, the moduli space $\mathcal{M}(1, n)$ is isomorphic to the Hilbert scheme of n points on $\mathbb{C}^2$ and $\mathcal{W}(\mathfrak{gl}_1)$ is the W-algebra associated with an infinite-dimensional Heisenberg algebra; the AGT correspondence for pure $U(1)$ gauge theory reduces to the famous result of Nakajima [46, 47] in the equivariant case [60, 36, 44]. Presently, (3) has been proved only in the rank one case [18] and in the rank two case [17, 49].

In this paper we are interested in the AGT correspondence for $\mathcal{N} = 2$ quiver gauge theories on ALE spaces associated with the Dynkin diagram of type A_{k-1} for $k \geq 2$. The corresponding instanton partition functions are defined in terms of equivariant cohomology classes over Nakajima quiver varieties of type the affine Dynkin diagram $\widehat{A}_{k-1}$. These quiver varieties depend on a real stability parameter $\xi_{\mathbb{R}}$, which lives in an open subset of $\mathbb{R}^k$ having a "chambers" decomposition: if two real stability parameters belong to the same chamber, the corresponding quiver varieties are (equivariantly) isomorphic; otherwise, the corresponding quiver varieties are only $\mathbb{C}^*$-diffeomorphic. Therefore, the pure gauge theories partition functions should be all nontrivially equivalent, while the partition functions for quiver gauge theories should satisfy "wall-crossing" formulas (cf. [7, 31]).

By looking at instanton partition functions of pure gauge theories associated with moduli spaces of $\mathbb{Z}_k$-equivariant framed sheaves on $\mathbb{P}^2$ (which are quiver varieties depending on a so-called "level zero chamber"), the authors of [9, 53, 6] conjectured an extension of the AGT correspondence in the A-type ALE case as a relation between instanton partition functions of $\mathcal{N} = 2$ quiver gauge theories and conformal blocks of Toda-like conformal field theories with $\mathbb{Z}_k$ parafermionic symmetry. In particular, the pertinent algebra to consider in this case is the coset

$$\mathcal{A}(r,k) := \frac{\widehat{\mathfrak{gl}}_N}{\widehat{\mathfrak{gl}}_{N-k}}$$

acting at level r, where N is related to the equivariant parameters. For $r = 1$ the algebra $\mathcal{A}(1,k)$ is simply $\widehat{\mathfrak{gl}}_k$ acting at level one. In general, $\mathcal{A}(r,k)$ is isomorphic to the direct sum of the affine Lie algebra $\widehat{\mathfrak{gl}}_k$ acting at level r and the $\mathbb{Z}_k$-parafermionic $\mathcal{W}(\mathfrak{gl}_r)$-algebra. Checks of the conjecture has been done [61, 30] by using partition functions of pure gauge theories associated with moduli spaces of $\mathbb{Z}_k$-equivariant framed sheaves on $\mathbb{P}^2$. In [10, 11] the authors studied in details $\mathcal{N} = 2$ quiver gauge theories on the minimal resolution X_2 of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_2$ and provided evidences for the conjecture: in this case, the quiver variety depends on a so-called "level infinity chamber" and corresponds to moduli spaces of framed sheaves on a suitable stacky compactification of X_2. In the $k = 2$ case, a comparison of these approaches using different stability chambers is done in [5]; further speculations in the arbitrary k case are in [12].

Mathematically, this correspondence should imply: (1) the existence of a representation of the coset $\mathcal{A}(r,k)$ on the equivariant cohomology of Nakajima quiver varieties associated with the affine A-type Dynkin diagram such that the latter is isomorphic to a Verma module of $\mathcal{A}(r,k)$; (2) the fundamental classes of the quiver varieties give a Whittaker vector of $\mathcal{A}(r,k)$ (pure gauge theory); (3) the Ext vertex operator is related to a certain "intertwiner" of $\mathcal{A}(r,k)$ under the isomorphism stated in (1) (quiver gauge theory). As pointed out in [5], different chambers should provide different realizations of the action conjectured in (1). On the other hand, the conjectural wall-crossing behavior of the instanton partition functions for quiver gauge theories [31] should be related by a similar behavior of the Ext vertex operators by varying of the stability chambers.

The ALE space we consider in this paper is the minimal resolution X_k of the simple Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_k$. In [14] an orbifold compactification $\mathscr{X}_k$ of X_k is constructed by adding a smooth divisor $\mathscr{D}_\infty$, which lays the foundations for a new sheaf theory approach to the study of $U(r)$ instantons on X_k (cf. [23]). Moduli spaces of sheaves on $\mathscr{X}_k$ framed along $\mathscr{D}_\infty$ are also constructed in [14]; by using these moduli spaces we have a new sheaf theory approach to the study of Nakajima quiver varieties with the stability parameter of X_k and, consequently, of $U(r)$ gauge theories on ALE spaces of type A_{k-1} which are isomorphic to X_k. In the present paper we use this new approach to study the AGT correspondence for abelian quiver gauge theories on X_k: from a physics point of view we prove the relations between instanton partition functions and conformal blocks and from a mathematical point of view we prove (1), (2) and (3).

1.2 Summary of results

Let us now summarize our main results. Recall that the compactification $\mathscr{X}_k$ is a two-dimensional projective toric orbifold with Deligne-Mumford torus $T := \mathbb{C}^* \times \mathbb{C}^*$; the complement $\mathscr{X}_k \setminus X_k$ is a smooth Cartier divisor $\mathscr{D}_\infty$ endowed with the structure of a $\mathbb{Z}_k$-gerbe. There exist line bundles $\mathcal{O}_{\mathscr{D}_\infty}(j)$ on $\mathscr{D}_\infty$, for $j = 0, 1, \ldots, k-1$, endowed with unitary flat connections associated with the irreducible unitary representations of $\mathbb{Z}_k$. Hence by [23, Theorem 6.9] locally free sheaves on $\mathscr{X}_k$ which are isomorphic along $\mathscr{D}_\infty$ to $\mathcal{O}_{\mathscr{D}_\infty}(j)$ correspond to $U(1)$ instantons on X_k with holonomy at infinity given by the j-th irreducible unitary representation of $\mathbb{Z}_k$, for $j = 0, 1, \ldots, k-1$.

Fix $j = 0, 1, \ldots, k-1$. A rank one $(\mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(j))$-framed sheaf on $\mathscr{X}_k$ is a pair $(\mathcal{E}, \phi_{\mathcal{E}})$, where $\mathcal{E}$ is a rank one torsion free sheaf on $\mathscr{X}_k$, locally free in a neighbourhood of $\mathscr{D}_\infty$, and $\phi_{\mathcal{E}} \colon \mathcal{E}\big|_{\mathscr{D}_\infty} \xrightarrow{\sim}$

$\mathcal{O}_{\mathscr{D}_\infty}(j)$ is an isomorphism. Let $\mathcal{M}(\vec{u}, n, j)$ be the fine moduli space parameterizing isomorphism classes of rank one $(\mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(j))$-framed sheaves on $\mathscr{X}_k$, with first Chern class given by $\vec{u} \in \mathbb{Z}^{k-1}$ and second Chern class n. As explained in Remark 5.11, the vector $\vec{u}$ is canonically associated with an element $\gamma_{\vec{u}} + \omega_j \in \mathfrak{Q} + \omega_j$, where $\mathfrak{Q}$ is the root lattice of the Dynkin diagram of type A_{k-1} and ω_j is the j-th fundamental weight of type A_{k-1}. We denote by $\mathfrak{U}_j$ the set of vectors $\vec{u}$ associated with $\gamma + \omega_j$ for some $\gamma \in \mathfrak{Q}$.

The moduli space $\mathcal{M}(\vec{u}, n, j)$ is a smooth quasi-projective variety of dimension $2n$. On $\mathcal{M}(\vec{u}, n; j)$ there is a natural T-action induced by the toric structure of $\mathscr{X}_k$. Let $\varepsilon_1, \varepsilon_2$ be the generators of the T-equivariant cohomology of a point and consider the localized equivariant cohomology

$$\mathbb{W}_{\vec{u},j} := \bigoplus_{n\geq 0} H_T^*\big(\mathcal{M}(\vec{u}, n, j)\big) \otimes_{\mathbb{C}[\varepsilon_1,\varepsilon_2]} \mathbb{C}(\varepsilon_1, \varepsilon_2) \ .$$

Define also the total localized equivariant cohomology by summing over all vectors $\vec{u} \in \mathfrak{U}_j$:

$$\mathbb{W}_j := \bigoplus_{\vec{u}\in\mathfrak{U}_j} \mathbb{W}_{\vec{u},j} \ .$$

The affine Lie algebra $\widehat{\mathfrak{gl}}_k$ acts on $\mathbb{W}_j$ as follows (see Proposition 6.24 and Proposition 6.28).

Proposition. *There exists a $\widehat{\mathfrak{gl}}_k$-action on $\mathbb{W}_j$ under which it is the j-th dominant representation of $\widehat{\mathfrak{gl}}_k$ at level one, i.e., the highest weight representation of $\widehat{\mathfrak{gl}}_k$ with fundamental weight $\widehat{\omega}_j$ of type $\widehat{A}_{k-1}$. Moreover, the weight spaces of $\mathbb{W}_j$ with respect to the $\widehat{\mathfrak{gl}}_k$-action are the $\mathbb{W}_{\vec{u},j}$ with weights $\gamma_{\vec{u}} + \omega_j$.*

The vector spaces $\mathbb{W}_{\vec{u},j}$ also have a representation theoretic intepretation.

Corollary (Corollary 6.27)**.** *$\mathbb{W}_{\vec{u},j}$ is a highest weight representation of the Virasoro algebra associated with $\widehat{\mathfrak{gl}}_k$ of conformal dimension $\Delta_{\vec{u}} := \frac{1}{2}\,\vec{u} \cdot C^{-1}\vec{u}$, where C is the Cartan matrix of type A_{k-1}.*

The representation is constructed by using a vertex algebra approach via the Frenkel-Kac construction. A similar construction for the cohomology groups of moduli spaces of rank one torsion free sheaves over smooth projective surfaces is outlined in [47, Chapter 9]. In [42], Nagao analysed vertex algebra realizations of representations of $\widehat{\mathfrak{sl}}_k$ on the equivariant cohomology groups of Nakajima quiver varieties associated with the affine Dynkin diagram $\widehat{A}_{k-1}$, for an integer $k \geq 2$, with dimension vector corresponding to the trivial holonomy at infinity $j = 0$; in this case the pertinent representation is the basic representation of $\widehat{\mathfrak{sl}}_k$.

In the following we describe our AGT relations, which connect together $\mathbb{W}_j$ for $j = 0, 1, \ldots, k-1$, the action of $\widehat{\mathfrak{gl}}_k$ on $\mathbb{W}_j$ and abelian quiver gauge theories on X_k. The first relation we obtain concerns the pure gauge theory. Let $\mathcal{Z}_{X_k}\big(\varepsilon_1, \varepsilon_2; \mathsf{q}, \vec{\xi}\,\big)_j$ be the instanton partition function for the pure $\mathcal{N} = 2\; U(1)$ gauge theory on the ALE space X_k with fixed holonomy at infinity given by the j-th irreducible representation of $\mathbb{Z}_k$ (see Section 8.1). It has the following representation theoretic characterization.

Theorem (AGT relation for pure $\mathcal{N} = 2\; U(1)$ gauge theory)**.** *The Gaiotto state*

$$G_j := \sum_{\vec{u}\in\mathfrak{U}_j} \sum_{n\geq 0} \big[\mathcal{M}(\vec{u}, n, j)\big]_T$$

is a Whittaker vector for $\widehat{\mathfrak{gl}}_k$. Moreover, the weighted norm of the weighted Gaiotto state

$$G_j(\mathsf{q},\vec{\xi}\,) := \sum_{\vec{u}\in\mathfrak{U}_j}\sum_{n\geq 0} \mathsf{q}^{n+\frac{1}{2}\,\vec{u}\cdot C^{-1}\vec{u}}\ \vec{\xi}^{\,C^{-1}\vec{u}}\ \big[\mathcal{M}(\vec{u},n,j)\big]_T$$

is exactly $\mathcal{Z}_{X_k}\big(\varepsilon_1,\varepsilon_2;\mathsf{q},\vec{\xi}\,\big)_j$.

We also consider $\mathcal{N}=2$ superconformal quiver gauge theories with gauge group $U(1)^{r+1}$ for some $r\geq 0$. By the ADE classification in [52, Chapter 3] the admissible quivers in this case are the linear quivers of the finite-dimensional A_r-type Dynkin diagram and the cyclic quivers of the affine $\widehat{A}_r$-type extended Dynkin diagram. In order to state AGT relations in these cases, we introduce Ext vertex operators [18, 17, 49]. Consider the element $\boldsymbol{E}_\mu\in K\big(\mathcal{M}(\vec{u}_1,n_1,j_1)\times\mathcal{M}(\vec{u}_2,n_2,j_2)\big)$ whose fibre over a point $\big([(\mathcal{E},\phi_{\mathcal{E}})],\,[(\mathcal{E}',\phi_{\mathcal{E}'})]\big)$ is

$$\big(\boldsymbol{E}_\mu\big)_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])} = \mathrm{Ext}^1\left(\mathcal{E},\mathcal{E}'\otimes\mathcal{O}_{\mathscr{X}_k}(\mu)\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\right)\,,$$

where $\mathcal{O}_{\mathscr{X}_k}(\mu)$ is the trivial line bundle on $\mathscr{X}_k$ on which the torus $T_\mu=\mathbb{C}^*$ acts by scaling the fibres with $H^*_{T_\mu}(\mathrm{pt};\mathbb{C})=\mathbb{C}[\mu]$. By using the Euler class of $\boldsymbol{E}_\mu$ we define a vertex operator $\mathrm{V}_\mu(\vec{x},z)\in\mathrm{End}\,\big(\bigoplus_{j=0}^{k-1}\mathbb{W}_j\,\big)[[z^{\pm 1},x_1^{\pm 1},\dots,x_{k-1}^{\pm 1}]]$ (see Section 7.1). Under the decomposition $\widehat{\mathfrak{gl}}_k=\mathfrak{h}\oplus\widehat{\mathfrak{sl}}_k$, we have the following characterization of $\mathrm{V}_\mu(\vec{x},z)$ in terms of vertex operators depending respectively on $\mathfrak{h}$ and $\widehat{\mathfrak{sl}}_k$.

Theorem (Theorem 7.6)**.** *The vertex operator $\mathrm{V}_\mu(\vec{x},z)$ can be expressed in the form*

$$\begin{aligned}\mathrm{V}_\mu(\vec{x},z) &= \mathrm{V}_{-\frac{\mu}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}},\frac{\mu+\varepsilon_1+\varepsilon_2}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}}}(z)\\ &\otimes\sum_{j_1,j_2=0}^{k-1}\ \sum_{\vec{u}_1\in\mathfrak{U}_{j_1},\vec{u}_2\in\mathfrak{U}_{j_2}}\bar{\mathrm{V}}_\mu(\vec{v}_{21},\vec{x},z)\,z^{\Delta_{\vec{u}_2}-\Delta_{\vec{u}_1}}\ \exp\big(\log z\ \mathfrak{c}-\gamma_{21}\,\big)\ \exp\big(\gamma_{21}\,\big)\big|_{\mathbb{W}_{\vec{u}_1,j_1}}\ ,\end{aligned}$$

where $\mathrm{V}_{\alpha,\beta}(z)$ denotes a generalized bosonic exponential associated with the Heisenberg algebra $\mathfrak{h}$ (see Definition 3.3), $\exp\big(\log z\ \mathfrak{c}-\gamma_{21}\,\big)\ \exp\big(\gamma_{21}\,\big)$ is the vertex operator on $\mathbb{W}_{j_1}$ defined in Equation (7.4)*, and $\bar{\mathrm{V}}_\mu(\vec{v}_{21},\vec{x},z)$ is the primary field* (7.5) *of the Virasoro algebra associated with $\widehat{\mathfrak{sl}}_k$ with conformal dimension $\Delta_{\vec{u}_2-\vec{u}_1}=\frac{1}{2}\,\vec{v}_{21}\cdot C\vec{v}_{21}$, where $\vec{v}_{21}:=C^{-1}(\vec{u}_2-\vec{u}_1)$ for $j_1,j_2=0,1,\dots,k-1$ and $\vec{u}_1\in\mathfrak{U}_{j_1},\vec{u}_2\in\mathfrak{U}_{j_2}$.*

For $j_1,j_2=0,1,\dots,k-1$ denote by $\mathrm{V}_\mu^{j_1,j_2}(\vec{x},z)$ the restriction of the vertex operator $\mathrm{V}_\mu(\vec{x},z)$ to $\mathrm{Hom}(\mathbb{W}_{j_1},\mathbb{W}_{j_2})[[z^{\pm 1},x_1^{\pm 1},\dots,x_{k-1}^{\pm 1}]]$.

Let $\mathcal{Z}_{X_k}^{\widehat{A}_r}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathsf{q},\vec{\boldsymbol{\xi}}\,\big)_{\boldsymbol{j}}$ be the instanton partition function for the $\mathcal{N}=2$ superconformal $U(1)^{r+1}$ quiver gauge theory of type $\widehat{A}_r$ with holonomy at infinity associated with $\boldsymbol{j}:=(j_0,j_1,\dots,j_r)$, topological couplings $\mathsf{q}_\upsilon\in\mathbb{C}^*$ and $\vec{\xi}_\upsilon\in(\mathbb{C}^*)^{k-1}$ for $\upsilon=0,1,\dots,r$, and masses $\boldsymbol{\mu}:=(\mu_0,\mu_1,\dots,\mu_r)$. We prove the following AGT relation.

Theorem (AGT relation for $\mathcal{N}=2$ $U(1)^{r+1}$ quiver gauge theory of type $\widehat{A}_r$)**.** *The partition function of the $\widehat{A}_r$-theory on X_k is given by*

$$\mathcal{Z}_{X_k}^{\widehat{A}_r}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathsf{q},\vec{\boldsymbol{\xi}}\,\big)_{\boldsymbol{j}} = \mathrm{Tr}_{\mathbb{W}_{j_0}}\ \mathsf{q}^{L_0}\ \vec{\xi}^{\,C^{-1}\vec{h}}\ \prod_{\upsilon=0}^{r}\ \mathrm{V}_{\mu_\upsilon}^{j_\upsilon,j_{\upsilon+1}}(\vec{x}_\upsilon,z_\upsilon)\ \delta_{\upsilon,\upsilon+1}^{\mathrm{conf}}\ ,$$

where $\mathsf{q} := \mathsf{q}_0\, \mathsf{q}_1 \cdots \mathsf{q}_r$, $(\vec{\xi}\,)_i := (\vec{\xi}_0)_i\, (\vec{\xi}_1)_i \cdots (\vec{\xi}_r)_i$, $z_\upsilon := z_0\, \mathsf{q}_1 \cdots \mathsf{q}_\upsilon$, *and* $(\vec{x}_\upsilon)_i := (\vec{x}_0)_i\, (\vec{\xi}_1)_i \cdots (\vec{\xi}_\upsilon)_i$ *for* $\upsilon = 1, \ldots, r$ *and* $i = 1, \ldots, k-1$*. Here* L_0 *is the Virasoro energy operator associated to* $\widehat{\mathfrak{gl}}_k$*,* $\vec{h} = (h_1, \ldots, h_{k-1})$ *are the generators of the Cartan subalgebra of* $\mathfrak{sl}_k$*, and* $\delta^{\mathrm{conf}}_{\upsilon,\upsilon+1}$ *is the conformal restriction operator defined in Equation* (8.6).

We also get a characterization of $\mathcal{Z}^{\widehat{A}_r}_{X_k}(\varepsilon_1, \varepsilon_2, \boldsymbol{\mu}; \mathbf{q}, \vec{\boldsymbol{\xi}}\,)_{\boldsymbol{j}}$ in terms of the corresponding partition function on $\mathbb{C}^2$ and a part depending only on $\widehat{\mathfrak{sl}}_k$.

Corollary. *Let* $\mathrm{V}_\mu(\vec{v}_{21}, \vec{x}, z) := z^{\Delta_{\vec{u}_2} - \Delta_{\vec{u}_1}}\, \bar{\mathrm{V}}_\mu(\vec{v}_{21}, \vec{x}, z)\ \exp\big(\log z\ \mathfrak{c} - \gamma_{21}\big)\ \exp\big(\gamma_{21}\big)$*. Then we have*

$$\mathcal{Z}^{\widehat{A}_r}_{X_k}(\varepsilon_1, \varepsilon_2, \boldsymbol{\mu}; \mathbf{q}, \vec{\boldsymbol{\xi}}\,)_{\boldsymbol{j}} = \mathcal{Z}^{\widehat{A}_r}_{\mathbb{C}^2}(\varepsilon_1, \varepsilon_2, \boldsymbol{\mu}; \mathsf{q})^{\frac{1}{k}}\ \mathsf{q}^{\frac{1}{24}(1-\frac{1}{k})}\, \eta(\mathsf{q})^{\frac{1}{k}-1} \times\ \mathrm{Tr}_{\mathcal{V}(\widehat{\omega}_{j_0})}\ \mathsf{q}^{L_0^{\widehat{\mathfrak{sl}}_k}}\ \vec{\xi}^{\,C^{-1}\vec{h}}\ \prod_{\upsilon=0}^{r}\ \sum_{(\vec{u}_\upsilon \in \mathfrak{U}^{\mathrm{conf}}_{j_\upsilon})} \mathrm{V}_{\mu_\upsilon}(\vec{v}_{\upsilon,\upsilon+1}, \vec{x}_\upsilon, z_\upsilon)\,\big|_{\mathbb{W}_{\vec{u}_\upsilon, j_\upsilon}}\ ,$$

where $\eta(\mathsf{q})$ *is the Dedekind function,* $\mathcal{V}(\widehat{\omega}_{j_0})$ *is the* j_0*-th dominant representation of* $\widehat{\mathfrak{sl}}_k$ *and* $\mathfrak{U}^{\mathrm{conf}}_{j_\upsilon}$ *is the subset of* $\mathfrak{U}_{j_\upsilon}$ *defined in Equation* (8.4).

Let $\mathcal{Z}^{A_r}_{X_k}(\varepsilon_1, \varepsilon_2, \boldsymbol{\mu}; \mathbf{q}, \vec{\boldsymbol{\xi}}\,)_{\boldsymbol{j}}$ be the instanton partition function for the $\mathcal{N} = 2$ superconformal $U(1)^{r+1}$ quiver gauge theory of type A_r with holonomy at infinity associated with $\boldsymbol{j} := (j_0, j_1, \ldots, j_r)$. We also prove the following AGT relation.

Theorem (AGT relation for $\mathcal{N} = 2\ U(1)^{r+1}$ quiver gauge theory of type A_r)**.** *The partition function of the* A_r*-theory on* X_k *is given by*

$$\mathcal{Z}^{A_r}_{X_k}(\varepsilon_1, \varepsilon_2, \boldsymbol{\mu}; \mathbf{q}, \vec{\boldsymbol{\xi}}\,)_{\boldsymbol{j}} = \Big\langle |0\rangle_{\mathrm{conf}}\ ,\ V_{\mu_0}(\vec{x}_0, z_0)\,\Big(\prod_{\upsilon=1}^{r} \mathrm{V}^{j_{\upsilon-1}, j_\upsilon}_{\mu_\upsilon}(\vec{x}_\upsilon, z_\upsilon)\ \delta^{\mathrm{conf}}_{\upsilon-1,\upsilon}\Big)\ \mathrm{V}_{\mu_{r+1}}(\vec{x}_{r+1}, z_{r+1})|0\rangle_{\mathrm{conf}}\Big\rangle_{\bigoplus_{j=0}^{k-1} \mathbb{W}_j}\ ,$$

where $z_\upsilon := z_0\, \mathsf{q}_0\, \mathsf{q}_1 \cdots \mathsf{q}_\upsilon$ *and* $(\vec{x}_\upsilon)_i := (\vec{x}_0)_i\, (\vec{\xi}_0)_i\, (\vec{\xi}_1)_i \cdots (\vec{\xi}_{\upsilon-1})_i$ *for* $\upsilon = 1, \ldots, r+1$*,* $i = 1, \ldots, k-1$*, and* $|0\rangle_{\mathrm{conf}} := \prod_{\upsilon=0}^{r} \delta^{\mathrm{conf}}_{0,\upsilon} \triangleright [\emptyset, \vec{0}\,]$ *with* $[\emptyset, \vec{0}\,]$ *the vacuum vector of the fixed point basis of* $\bigoplus_{j=0}^{k-1} \mathbb{W}_j$*.*

Denote by $\mathcal{V}$ the direct sum of the k level one dominant representations of $\widehat{\mathfrak{sl}}_k$. Similarly to before, we have the following characterization.

Corollary. *We have*

$$\mathcal{Z}^{A_r}_{X_k}(\varepsilon_1, \varepsilon_2, \boldsymbol{\mu}; \mathbf{q}, \vec{\boldsymbol{\xi}}\,)_{\boldsymbol{j}} = \mathcal{Z}^{A_r}_{\mathbb{C}^2}(\varepsilon_1, \varepsilon_2, \boldsymbol{\mu}; \mathbf{q})^{\frac{1}{k}} \times\ \Big\langle |0\rangle_{\mathrm{conf}}\ ,\ \Big(\sum_{j_0, j_0'=0}^{k-1}\ \sum_{\vec{u}_0 \in \mathfrak{U}_{j_0}, \vec{u}_0' \in \mathfrak{U}_{j_0'}} \mathrm{V}_{\mu_0}(\vec{v}_{0',0}, \vec{x}_0, z_0)\,\big|_{\mathbb{W}_{\vec{u}_0, j_0}}\Big) \times \prod_{\upsilon=1}^{r}\ \sum_{(\vec{u}_\upsilon \mathfrak{U}^{\mathrm{conf}}_{j_\upsilon})} \mathrm{V}_{\mu_\upsilon}(\vec{v}_{\upsilon-1,\upsilon}, \vec{x}_\upsilon, z_\upsilon)\,\big|_{\mathbb{W}_{\vec{u}_\upsilon, j_\upsilon}}$$

$$\times \Big(\sum_{j_{r+1}, j'_{r+1}=0}^{k-1} \ \sum_{\vec{u}_1 \in \mathfrak{U}_{j_1}, \vec{u}'_1 \in \mathfrak{U}_{j'_1}} \mathrm{V}_{\mu_{r+1}}(\vec{v}_{1',1}, \vec{x}_{r+1}, z_{r+1}) \big|_{\mathbb{W}_{\vec{u}_1, j_1}} \Big) |0\rangle_{\mathrm{conf}} \Big\rangle_{\mathcal{V}} .$$

Another important aspect of the AGT correspondence that we address in this paper is the relation of our construction with quantum integrable systems. In particular, for any $j = 0, 1, \ldots, k-1$ we define an infinite system of commuting operators which are diagonalized in the fixed point basis of $\mathbb{W}_j$; geometrically these operators correspond to multiplication by equivariant cohomology classes (see Section 7.3). The eigenvalues of these operators with respect to this basis can be decomposed into a part associated with k non-interacting Calogero-Sutherland models and a part which can be interpreted as particular matrix elements of the vertex operators $\mathrm{V}_\mu(\vec{x}, z)$ in highest weight vectors of $\widehat{\mathfrak{gl}}_k$. The significance of this property is that this special orthogonal basis manifests itself in the special integrable structure of the two-dimensional conformal field theory and yields completely factorized matrix elements of composite vertex operators explicitly in terms of simple rational functions of the basic parameters, which from the gauge theory perspective represent the contributions of bifundamental matter fields.

The study of the AGT relation for pure $\mathcal{N} = 2$ $U(1)$ gauge theories and the problem of constructing commuting operators associated with $\widehat{\mathfrak{gl}}_k$ is also addressed in [8] from another point of view: there they consider the "conformal" limit of the Ding-Iohara algebra, depending on parameters q, t, for q, t approaching a primitive k-th root of unity and relate the representation theory of this limit to the AGT correspondence. However, their point of view is completely algebraic, so unfortunately it is not clear to us how to geometrically construct the action of the conformal limit on the equivariant cohomology groups.

1.3 Outline

This paper is structured as follows. In Section 2 we briefly recall the relevant combinatorial notions that we use in this paper. In Section 3 we collect preliminary material on Heisenberg algebras and affine Lie algebras of type $\widehat{A}_{k-1}$, giving particular attention to the Frenkel-Kac construction of level one dominant representations of $\widehat{\mathfrak{sl}}_k$ and $\widehat{\mathfrak{gl}}_k$. In Section 4 we review the AGT relations for $\mathcal{N} = 2$ superconformal abelian quiver gauge theories on $\mathbb{R}^4$. In Section 5 we briefly recall the construction of the orbifold compactification $\mathscr{X}_k$ and of moduli spaces of framed sheaves on $\mathscr{X}_k$ from [14]. Section 6 addresses the construction of the action of $\widehat{\mathfrak{gl}}_k$ on $\mathbb{W}_j$ for $j = 0, 1, \ldots, k-1$: we perform a vertex algebra construction of the representation by using the Frenkel-Kac theorem. In Section 7 we define the virtual bundle $\boldsymbol{E}_\mu$ and the vertex operator $\mathrm{V}_\mu(\vec{x}, z)$, and we characterize it in terms of vertex operators of an infinite-dimensional Heisenberg algebra $\mathfrak{h}$ and primary fields of $\widehat{\mathfrak{sl}}_k$ under the decomposition $\widehat{\mathfrak{gl}}_k = \mathfrak{h} \oplus \widehat{\mathfrak{sl}}_k$; moreover, we geometrically define an infinite system of commuting operators. In Section 8 we prove our AGT relations, and furthermore provide expressions for our partition functions in terms of the corresponding partition functions on $\mathbb{C}^2$ and a part depending only on $\widehat{\mathfrak{sl}}_k$. The paper concludes with two Appendices containing some technical details of the constructions from the main text: in Appendix A we give the proof that the vertex operator $\bar{\mathrm{V}}_\mu(\vec{v}_{21}, \vec{x}, z)$ is a primary field, while in Appendix B we recall the expressions from [14] for the edge factors which appear in the definition of $\bar{\mathrm{V}}_\mu(\vec{v}_{21}, \vec{x}, z)$ as well as in the eigenvalues of the integrals of motion.

1.4 Acknowledgements

We are grateful to M. Bershtein, A. Konechny, O. Schiffmann and E. Vasserot for helpful discussions. Also, we are indebted to the anonymous referee, whose remarks helped to improve the paper. This work was supported in part by PRIN "Geometria delle varietà algebriche", by GNSAGA-INDAM, by the Grant RPG-404 from the Leverhulme Trust, and by the Consolidated Grant ST/J000310/1 from the UK Science and Technology Facilities Council. The bulk of this paper was written while the authors were staying at Heriot-Watt University in Edinburgh and at SISSA in Trieste. The last draft of the paper was written while the first and second authors were staying at IHP in Paris under the auspices of the RIP program. We thank these institutions for their hospitality and support.

2 Combinatorial preliminaries

2.1 Partitions and Young tableaux

A *partition* of a positive integer n is a nonincreasing sequence of positive numbers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ such that $|\lambda| := \sum_{a=1}^{\ell} \lambda_a = n$. We call $\ell = \ell(\lambda)$ the *length* of the partition λ. Another description of a partition λ of n uses the notation $\lambda = (1^{m_1}\, 2^{m_2}\, \cdots)$, where $m_i = \#\{a \in \mathbb{N} \,|\, \lambda_a = i\}$ with $\sum_i i\, m_i = n$ and $\sum_i m_i = \ell(\lambda)$. On the set of all partitions there is a natural partial ordering called *dominance ordering*: For two partitions μ and λ, we write $\mu \leq \lambda$ if and only if $|\mu| = |\lambda|$ and $\mu_1 + \cdots + \mu_a \leq \lambda_1 + \cdots + \lambda_a$ for all $a \geq 1$. We write $\mu < \lambda$ if and only if $\mu \leq \lambda$ and $\mu \neq \lambda$.

One can associate with a partition λ its *Young tableau*, which is the set $Y_\lambda = \{(a, b) \in \mathbb{N}^2 \,|\, 1 \leq a \leq \ell(\lambda)\,,\, 1 \leq b \leq \lambda_a\}$. Then λ_a is the length of the a-th column of Y_λ; we write $|Y_\lambda| = |\lambda|$ for the *weight* of the Young tableau Y_λ. We shall identify a partition λ with its Young tableau Y_λ. For a partition λ, the *transpose partition* λ' is the partition whose Young tableau is $Y_{\lambda'} := \{(b, a) \in \mathbb{N}^2 \,|\, (a, b) \in Y_\lambda\}$.

The elements of a Young tableau Y are called the *nodes* of Y. For a node $s = (a, b) \in Y$, the *arm length* of s is the quantity $A(s) := A_Y(s) = \lambda_a - b$ and the *leg length* of s the quantity $L(s) := L_Y(s) = \lambda'_b - a$. The *arm colength* and *leg colength* are respectively given by $A'(s) := A'_Y(s) = b-1$ and $L'(s) := L'_Y(s) = a - 1$.

2.2 Symmetric functions

Here we recall some preliminaries about the theory of symmetric functions in infinitely many variables which we shall use later on. Our main reference is [37].

Let $\mathbb{F}$ be a field of characteristic zero. The *algebra of symmetric polynomials in N variables* is the subspace $\Lambda_{\mathbb{F},N}$ of $\mathbb{F}[x_1, \ldots, x_N]$ which is invariant under the action of the group of permutations σ_N on N letters. Then $\Lambda_{\mathbb{F},N}$ is a graded ring: $\Lambda_{\mathbb{F},N} = \bigoplus_{n\geq 0} \Lambda^n_{\mathbb{F},N}$, where $\Lambda^n_{\mathbb{F},N}$ is the ring of homogeneous symmetric polynomials in N variables of degree n (together with the zero polynomial).

For any $M > N$ there are morphisms $\rho_{MN} \colon \Lambda_{\mathbb{F},M} \to \Lambda_{\mathbb{F},N}$ that map the variables $x_{N+1}, \ldots, x_M$ to zero. They preserve the grading, and hence we can define $\rho^n_{MN} \colon \Lambda^n_{\mathbb{F},M} \to \Lambda^n_{\mathbb{F},N}$; this allows us to define the inverse limits

$$\Lambda^n_{\mathbb{F}} := \lim_{\substack{\longleftarrow \\ N}} \Lambda^n_{\mathbb{F},N}\ ,$$

and the *algebra of symmetric functions in infinitely many variables* as $\Lambda_{\mathbb{F}} := \bigoplus_{n\geq 0} \Lambda^n_{\mathbb{F}}$. In the following when no confusion is possible we will denote $\Lambda_{\mathbb{F}}$ (resp. $\Lambda^n_{\mathbb{F}}$) simply by Λ (resp. Λ^n).

Now we introduce a basis for Λ. For this, we start by defining a basis in Λ_N. Let $\mu = (\mu_1, \ldots, \mu_t)$ be a partition with $t \leq N$, and define the polynomial

$$m_\mu(x_1, \ldots, x_N) = \sum_{\tau\in\sigma_N} x_1^{\mu_{\tau(1)}} \cdots x_N^{\mu_{\tau(N)}} ,$$

where we set $\mu_j = 0$ for $j = t+1, \ldots, N$. The polynomial m_μ is symmetric, and the set of m_μ for all partitions μ with $|\mu| \leq N$ is a basis of Λ_N. Then the set of m_μ, for all partitions μ with $|\mu| \leq N$ and $\sum_i \mu_i = n$, is a basis of Λ^n_N. Since for $M > N \geq t$ we have $\rho^n_{MN}(m_\mu(x_1, \ldots, x_M)) = m_\mu(x_1, \ldots, x_N)$, by using the definition of inverse limit we can define the *monomial symmetric functions* m_μ. By varying over the partitions μ of n, these functions form a basis for Λ^n.

Next we define the *n-th power sum symmetric function* p_n as

$$p_n := m_{(n)} = \sum_i x_i^n .$$

The set consisting of symmetric functions $p_\mu := p_{\mu_1} \ldots p_{\mu_t}$, for all partitions $\mu = (\mu_1, \ldots, \mu_t)$, is another basis of Λ.

We now set $\mathbb{F} = \mathbb{C}$ throughout and we fix a parameter $\beta \in \mathbb{C}$ (though everything works for any field extension $\mathbb{C} \subseteq \mathbb{F}$ and $\beta \in \mathbb{F}$). Define an inner product $\langle -, - \rangle_\beta$ on the vector space $\Lambda \otimes \mathbb{Q}(\beta)$ with respect to which the basis of power sum symmetric functions $p_\lambda(x)$ are orthogonal with the normalization

$$\langle p_\lambda, p_\mu \rangle_\beta = \delta_{\lambda,\mu}\, z_\lambda\, \beta^{-\ell(\lambda)} , \tag{2.1}$$

where $\delta_{\lambda,\mu} := \prod_a \delta_{\lambda_a,\mu_a}$ and

$$z_\lambda := \prod_{j\geq 1} j^{m_j}\, m_j! .$$

This is called the *Jack inner product*.

Definition 2.2. The monic forms of the *Jack functions* $J_\lambda(x; \beta^{-1}) \in \Lambda \otimes \mathbb{Q}(\beta)$ for $x = (x_1, x_2, \ldots)$ are uniquely defined by the following two conditions [37]:

(i) Triangular expansion in the basis $m_\mu(x)$ of monomial symmetric functions:

$$J_\lambda(x; \beta^{-1}) = m_\lambda(x) + \sum_{\mu<\lambda} \psi_{\lambda,\mu}(\beta)\, m_\mu(x) \qquad \text{with} \quad \psi_{\lambda,\mu}(\beta) \in \mathbb{Q}(\beta) .$$

(ii) Orthogonality:

$$\langle J_\lambda, J_\mu \rangle_\beta = \delta_{\lambda,\mu} \prod_{s\in Y_\lambda} \frac{\beta\, L(s) + A(s) + 1}{\beta\, \big(L(s) + 1\big) + A(s)} .$$

⊘

Lemma 2.3. *For any integer $n \geq 1$ we have*

$$(p_1)^n = n! \sum_{|\lambda|=n} \prod_{s\in Y_\lambda} \frac{1}{\beta\, L(s) + A(s) + 1}\, J_\lambda .$$

Proof. The assertion follows straightforwardly from [58, Proposition 2.3 and Theorem 5.8] after normalizing our Jack functions: the Jack functions considered in [58] are given by

$$\tilde{J}_\lambda = \beta^{-|\lambda|} \prod_{s\in Y_\lambda} \left[\beta\left(L(s)+1\right)+A(s)\right] J_\lambda \, ,$$

where the normalization factor is computed by using [58, Theorem 5.6]. □

3 Infinite-dimensional Lie algebras

3.1 Heisenberg algebras

In this section we recall the representation theory of Heisenberg algebras and the affine Lie algebras $\widehat{\mathfrak{sl}}_k$. Since the Lie algebra $\mathfrak{gl}_k$ coincides with $\mathbb{F}\,\mathrm{id}\oplus\mathfrak{sl}_k$, as a by-product we get the representation theory of $\widehat{\mathfrak{gl}}_k$.

Let $\mathbb{C}\subseteq\mathbb{F}$ be a field extension of $\mathbb{C}$. Let $\mathfrak{L}$ be a lattice, i.e., a free abelian group of finite rank d equipped with a symmetric nondegenerate bilinear form $\langle -,-\rangle_{\mathfrak{L}}\colon \mathfrak{L}\times\mathfrak{L}\to\mathbb{Z}$. Fix a basis $\gamma_1,\ldots,\gamma_d$ of $\mathfrak{L}$.

Definition 3.1. The *lattice Heisenberg algebra* $\mathfrak{h}_{\mathbb{F},\mathfrak{L}}$ associated with $\mathfrak{L}$ is the infinite-dimensional Lie algebra over $\mathbb{F}$ generated by $\mathfrak{q}^i_m$, for $m\in\mathbb{Z}\setminus\{0\}$ and $i\in\{1,\ldots,d\}$, and the central element $\mathfrak{c}$ satisfying the relations

$$\begin{cases} \left[\mathfrak{q}^i_m,\mathfrak{c}\right]=0 & \text{for} \quad m\in\mathbb{Z}\setminus\{0\}\ ,\ i\in\{1,\ldots,d\}\ , \\ \left[\mathfrak{q}^i_m,\mathfrak{q}^j_n\right]=m\,\delta_{m,-n}\,\langle\gamma_i,\gamma_j\rangle_{\mathfrak{L}}\,\mathfrak{c} & \text{for} \quad m,n\in\mathbb{Z}\setminus\{0\}\ ,\ i,j\in\{1,\ldots,d\}\ . \end{cases} \tag{3.2}$$

⊘

For any element $v\in\mathfrak{L}$ we define the element $\mathfrak{q}^v_m\in\mathfrak{h}_{\mathbb{F},\mathfrak{L}}$ by linearity, with $\mathfrak{q}^i_m:=\mathfrak{q}^{\gamma_i}_m$. Set

$$\mathfrak{h}^+_{\mathbb{F},\mathfrak{L}}:=\bigoplus_{m>0}\bigoplus_{i=1}^d\mathbb{F}\mathfrak{q}^i_m \qquad\text{and}\qquad \mathfrak{h}^-_{\mathbb{F},\mathfrak{L}}:=\bigoplus_{m<0}\bigoplus_{i=1}^d\mathbb{F}\mathfrak{q}^i_m\ .$$

Let us denote by $\mathcal{U}(\mathfrak{h}_{\mathbb{F},\mathfrak{L}})$ (resp. $\mathcal{U}(\mathfrak{h}^\pm_{\mathbb{F},\mathfrak{L}})$) the universal enveloping algebra of $\mathfrak{h}_{\mathbb{F},\mathfrak{L}}$ (resp. $\mathfrak{h}^\pm_{\mathbb{F},\mathfrak{L}}$), i.e., the unital associative algebra over $\mathbb{F}$ generated by $\mathfrak{h}_{\mathbb{F},\mathfrak{L}}$ (resp. $\mathfrak{h}^\pm_{\mathbb{F},\mathfrak{L}}$).

We introduce some terminology similar to that used in [25, Section 5.2.5].

Definition 3.3. For $v\in\mathfrak{L}$, define *free bosonic fields* as the elements

$$\varphi^v_-(z):=\sum_{m=1}^\infty\frac{z^m}{m}\,\mathfrak{q}^v_{-m} \qquad\text{and}\qquad \varphi^v_+(z):=\sum_{m=1}^\infty\frac{z^{-m}}{m}\,\mathfrak{q}^v_m$$

in $\mathfrak{h}^-_{\mathbb{F},\mathfrak{L}}[[z]]$ and $\mathfrak{h}^+_{\mathbb{F},\mathfrak{L}}[[z^{-1}]]$, respectively. For $\alpha,\beta\in\mathbb{F}$, define the *generalized bosonic exponential*

$$\mathrm{V}^v_{\alpha,\beta}(z):=\exp\left(\alpha\,\varphi^v_-(z)\right)\,\exp\left(\beta\,\varphi^v_+(z)\right)=:\ :\ \exp\left(\alpha\,\varphi_-(z)+\beta\,\varphi_+(z)\right)\ :$$

in $\mathfrak{h}_{\mathbb{F},\mathfrak{L}}[[z,z^{-1}]]$, where the symbol $:-:$ denotes normal ordering with respect to the decomposition $\mathfrak{h}_{\mathbb{F},\mathfrak{L}}=\mathfrak{h}^-_{\mathbb{F},\mathfrak{L}}\oplus\mathfrak{h}^+_{\mathbb{F},\mathfrak{L}}$, i.e., all negative generators $\mathfrak{q}^v_{-m}$ are moved to the left of all positive generators $\mathfrak{q}^v_m$ for $m>0$. When $\beta=-\alpha$, we call $\mathrm{V}^v_{\alpha,-\alpha}(z)$ a *normal-ordered bosonic exponential*. ⊘

Remark 3.4. The bosonic exponentials are *vertex operators*, i.e., they are uniquely characterized by their commutation relations in the Heisenberg algebra $\mathfrak{h}_{\mathbb{F},\mathfrak{L}}$: For $v, v' \in \mathfrak{L}$ one has

$$\big[\mathfrak{q}_m^v, \mathrm{V}_{\alpha,\beta}^{v'}(z)\big] = \begin{cases} \alpha \, \langle v, v' \rangle_{\mathfrak{L}} \, z^m \, \mathrm{V}_{\alpha,\beta}^{v'}(z) & \text{for} \quad m > 0 \, , \\ -\beta \, \langle v, v' \rangle_{\mathfrak{L}} \, z^m \, \mathrm{V}_{\alpha,\beta}^{v'}(z) & \text{for} \quad m < 0 \, . \end{cases}$$

The compositions of vertex operators $\mathrm{V}_{\alpha_1,\beta_1}^{v_1}(z_1) \cdots \mathrm{V}_{\alpha_n,\beta_n}^{v_n}(z_n)$ in $\mathfrak{h}_{\mathbb{F},\mathfrak{L}}[[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]]$ can be easily calculated as

$$\prod_{i=1}^{n} \mathrm{V}_{\alpha_i,\beta_i}^{v_i}(z_i) = \Big(\prod_{1 \leq j < i \leq n} \Big(1 - \frac{z_i}{z_j}\Big)^{-\alpha_i \, \beta_j \, \langle v_i, v_j \rangle_{\mathfrak{L}}} \Big) \, : \prod_{i=1}^{n} \mathrm{V}_{\alpha_i,\beta_i}^{v_i}(z_i) \, : \, , \tag{3.5}$$

where the factors $(1 - \frac{z_i}{z_j})^{-\alpha_i \, \beta_j \, \langle v_i, v_j \rangle_{\mathfrak{L}}}$ are understood as formal power series in $\frac{z_i}{z_j}$. △

Remark 3.6. When $v = \gamma_i$ for $i = 1, \ldots, d$, we simply denote $\varphi_\pm^i(z) := \varphi_\pm^{\gamma_i}(z)$; if $d = 1$, we further simply write $\varphi_\pm(z)$. We use analogous notation for the generalized free boson exponentials. △

Example 3.7. Consider the lattice $\mathfrak{L} := \mathbb{Z}^k$ with the symmetric nondegenerate bilinear form $\langle v, w \rangle_{\mathfrak{L}} = \sum_{i=1}^k v_i \, w_i$. In this case $\mathfrak{h}_{\mathbb{F},\mathfrak{L}}$ is called the *Heisenberg algebra of rank k* over $\mathbb{F}$, and we denote it by $\mathfrak{h}_{\mathbb{F}}^k$. It is generated by elements $\mathfrak{p}_m^i$, $m \in \mathbb{Z} \setminus \{0\}$, $i = 1, \ldots, k$, and the central element $\mathfrak{c}$ satisfying the relations (3.2) with $\langle \gamma_i, \gamma_j \rangle_{\mathfrak{L}} = \delta_{ij}$. When $k = 1$, $\mathfrak{h}_{\mathbb{F},\mathfrak{L}}$ is simply the infinite-dimensional *Heisenberg algebra* $\mathfrak{h}_{\mathbb{F}}$ over the field $\mathbb{F}$.

Example 3.8. Fix an integer $k \geq 2$ and let $\mathfrak{Q}$ be the root lattice of type A_{k-1} endowed with the standard bilinear form $\langle -, - \rangle_{\mathfrak{Q}}$ (see Remark 3.16 below). Let $\mathfrak{h}_{\mathbb{F},\mathfrak{Q}}$ be the lattice Heisenberg algebra over $\mathbb{F}$ associated to $\mathfrak{Q}$; we call $\mathfrak{h}_{\mathbb{F},\mathfrak{Q}}$ the *Heisenberg algebra of type A_{k-1}* over $\mathbb{F}$. It can be realized as the Lie algebra over $\mathbb{F}$ generated by $\mathfrak{q}_m^i$ for $m \in \mathbb{Z} \setminus \{0\}$, $i = 1, \ldots, k-1$, and the central element $\mathfrak{c}$ satisfying the relations

$$\begin{cases} \big[\mathfrak{q}_m^i, \mathfrak{c}\big] = 0 & \text{for} \quad m \in \mathbb{Z} \setminus \{0\} \, , \; i = 1, \ldots, k-1 \, , \\ \big[\mathfrak{q}_m^i, \mathfrak{q}_n^j\big] = m \, \delta_{m,-n} \, C_{ij} \, \mathfrak{c} & \text{for} \quad m \in \mathbb{Z} \setminus \{0\} \, , \; i, j = 1, \ldots, k-1 \, , \end{cases}$$

where $C = (C_{ij})$ is the Cartan matrix of type A_{k-1}.

3.1.1 Virasoro generators

We construct the Viraroso algebra associated with the Heisenberg algebra $\mathfrak{h}_{\mathbb{F}}$. Define elements

$$L_0^{\mathfrak{h}} = \sum_{m=1}^{\infty} \mathfrak{q}_{-m} \, \mathfrak{q}_m \qquad \text{and} \qquad L_n^{\mathfrak{h}} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \mathfrak{q}_{-m} \, \mathfrak{q}_{m+n} \quad \text{for} \quad n \in \mathbb{Z} \setminus \{0\}$$

in the completion of the enveloping algebra $\mathcal{U}(\mathfrak{h}_{\mathbb{F}})$, where we set $\mathfrak{q}_0 := 0$. They satisfy the relations

$$\big[L_n^{\mathfrak{h}}, L_m^{\mathfrak{h}}\big] = (n - m) \, L_{n+m}^{\mathfrak{h}} + \frac{n}{12} \, (n^2 - 1) \, \delta_{m+n,0} \, \mathfrak{c} \, ,$$

hence $\mathfrak{c}$ and $L_n^{\mathfrak{h}}$ with $n \in \mathbb{Z}$ generate a Virasoro algebra $\mathfrak{Vir}_{\mathbb{F}}$ over $\mathbb{F}$.

Remark 3.9. It is well-known (see Appendix A) that the generalized bosonic exponential $\mathrm{V}_{\alpha,\beta}(z)$ is a primary field of the Virasoro algebra $\mathfrak{Vir}_{\mathbb{F}}$ generated by $L_n^{\mathfrak{h}}$ with conformal dimension $\Delta(\alpha, \beta) = -\frac{1}{2} \, \alpha \, \beta$, i.e., it satisfies the commutation relations

$$\big[L_n^{\mathfrak{h}}, \mathrm{V}_{\alpha,\beta}(z)\big] = z^n \, \big(z \, \partial_z + \Delta(\alpha, \beta) \, (n + 1)\big) \, \mathrm{V}_{\alpha,\beta}(z) \, .$$

△

3.1.2 Fock space

We are interested in a special type of representation of a given lattice Heisenberg algebra $\mathfrak{h}_{\mathbb{F},\mathfrak{L}}$ over $\mathbb{F}$.

Definition 3.10. Let W be the trivial representation of $\mathfrak{h}^+_{\mathbb{F},\mathfrak{L}}$ (i.e., the one-dimensional $\mathbb{F}$-vector space with trivial $\mathfrak{h}^+_{\mathbb{F},\mathfrak{L}}$-action). The *Fock space* representation of the Heisenberg algebra $\mathfrak{h}_{\mathbb{F},\mathfrak{L}}$ is the induced representation $\mathcal{F}_{\mathbb{F},\mathfrak{L}} := \mathfrak{h}_{\mathbb{F},\mathfrak{L}} \otimes_{\mathfrak{h}^+_{\mathbb{F},\mathfrak{L}}} W$. $\oslash$

The Fock space is an irreducible *highest weight representation* whereby any element $w_0 \in W$ is a *highest weight vector*, i.e., $\mathfrak{h}^+_{\mathbb{F},\mathfrak{L}}$ annihilates w_0 and the elements in W of the form $\mathfrak{q}^v_{-m_1} \cdots \mathfrak{q}^v_{-m_l} \triangleright w_0$ generate $\mathcal{F}_{\mathbb{F},\mathfrak{L}}$ for $v \in \mathfrak{L}$, $l \geq 1$ and $m_i \geq 1$ for $i = 1, \dots, l$.

Example 3.11. For the Heisenberg algebra $\mathfrak{h}_{\mathbb{F}}$, the Fock space $\mathcal{F}_{\mathbb{F}}$ is isomorphic to the polynomial algebra $\Lambda_{\mathbb{F}} = \mathbb{F}[p_1, p_2, \dots]$ in the power sum symmetric functions introduced in Section 2.2. In this realization, the actions of the generators are given for $m > 0$ by

$$\mathfrak{p}_{-m} \triangleright f := p_m f \ , \qquad \mathfrak{p}_m \triangleright f := m \, \frac{\partial f}{\partial p_m} \qquad \text{and} \qquad \mathfrak{c} \triangleright f := f \tag{3.12}$$

for any $f \in \Lambda_{\mathbb{F}}$.

Example 3.13. The Fock space $\mathcal{F}^k_{\mathbb{F}}$ of the rank k Heisenberg algebra $\mathfrak{h}^k_{\mathbb{F}}$ can be realized as the tensor product of k copies of the polynomial algebra $\Lambda_{\mathbb{F}}$:

$$\mathcal{F}^k_{\mathbb{F}} \simeq \Lambda^{\otimes k}_{\mathbb{F}} \ .$$

In this realization, the action of the generators $\mathfrak{p}^i_m$ is obvious: each copy of the Heisenberg algebra generated by $\mathfrak{p}^i_m$ for $m \in \mathbb{Z} \setminus \{0\}$ acts on the i-th factor $\Lambda_{\mathbb{F}}$ as in Equation (3.12).

3.1.3 Whittaker vectors

We give the definition of Whittaker vector for Heisenberg algebras following [19, Section 3]; in conformal field theory it has the meaning of a *coherent state*.

Definition 3.14. Let $\chi \colon \mathcal{U}(\mathfrak{h}^+_{\mathbb{F},\mathfrak{L}}) \to \mathbb{F}$ be an algebra homomorphism such that $\chi|_{\mathfrak{h}^+_{\mathbb{F},\mathfrak{L}}} \neq 0$, and let V be a $\mathcal{U}(\mathfrak{h}_{\mathbb{F},\mathfrak{L}})$-module. A nonzero vector $w \in V$ is called a *Whittaker vector of type* χ if $\eta \triangleright w = \chi(\eta)\, w$ for all $\eta \in \mathcal{U}(\mathfrak{h}^+_{\mathbb{F},\mathfrak{L}})$. $\oslash$

Remark 3.15. By [19, Proposition 10], if w, w' are Whittaker vectors of the same type χ, then $w' = \lambda\, w$ for some nonzero $\lambda \in \mathbb{F}$. $\triangle$

3.2 Affine algebra of type $\widehat{A}_{k-1}$

Let $k \geq 2$ be an integer and let $\mathfrak{sl}_k := \mathfrak{sl}(k, \mathbb{F})$ denote the finite-dimensional Lie algebra of rank $k-1$ over $\mathbb{F}$ generated in the Chevalley basis by E_i, F_i, H_i for $i = 1, \dots, k-1$ satisfying the relations

$$\begin{aligned} [E_i, F_j] &= \delta_{ij}\, H_j \ , \quad [H_i, H_j] = 0 \ , \\ [H_i, E_j] &= C_{ij}\, E_j \ , \quad [H_i, F_j] = -C_{ij}\, F_j \ , \end{aligned}$$

where $C = (C_{ij})$ is the Cartan matrix type A_{k-1} (see Remark 3.16 below).

An explicit realization of the generators of $\mathfrak{sl}_k$ in the algebra $M(k,\mathbb{F})$ of $k \times k$ matrices over $\mathbb{F}$ is given in the following way. Let $\boldsymbol{E}_{i,j}$ denote the $k \times k$ matrix unit with 1 in the (i,j) entry and 0 everywhere else for $i,j = 1,\dots,k$. Define

$$E_i := \boldsymbol{E}_{i,i+1} \ , \qquad F_i := \boldsymbol{E}_{i+1,i} \qquad \text{and} \qquad H_i := \boldsymbol{E}_{i,i} - \boldsymbol{E}_{i+1,i+1}$$

for $i = 1,\dots,k-1$. One sees immediately that E_i, F_i, H_i satisfy the defining relations for $\mathfrak{sl}_k$.

Let us denote by $\mathfrak{t}$ the Lie subalgebra of $\mathfrak{sl}_k$ generated by H_i for $i = 1,\dots,k-1$ and by $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) the Lie subalgebra of $\mathfrak{sl}_k$ generated by E_i (resp. F_i) for $i = 1,\dots,k-1$. Then there is a triangular decomposition

$$\mathfrak{sl}_k = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+$$

as a direct sum of vector spaces.

Remark 3.16. For $i = 1,\dots,k$, define $\boldsymbol{e}_i \in \mathfrak{t}^*$ by

$$\boldsymbol{e}_i\big(\mathrm{diag}(a_1,\dots,a_k)\big) = a_i \ .$$

The elements $\gamma_i := \boldsymbol{e}_i - \boldsymbol{e}_{i+1}$ for $i = 1,\dots,k-1$ form a basis of $\mathfrak{t}^*$. The *root lattice* $\mathfrak{Q}$ is the lattice $\mathfrak{Q} := \bigoplus_{i=1}^{k-1} \mathbb{Z}\gamma_i$. The elements of $\mathfrak{Q}$ are called *roots*, and in particular γ_i are called the *simple roots*. The lattice of positive roots is $\mathfrak{Q}_+ := \bigoplus_{i=1}^{k-1} \mathbb{N}\gamma_i$. Since $\boldsymbol{e}_i$ corresponds to the i-th coordinate vector in $\mathbb{Z}^k$, there is a description of $\mathfrak{Q}$ and $\mathfrak{Q}_+$ in $\mathbb{Z}^k$ given by

$$\mathfrak{Q} = \big\{\boldsymbol{e}_i - \boldsymbol{e}_j \,\big|\, i,j = 1,\dots,k\big\} \qquad \text{and} \qquad \mathfrak{Q}_+ = \big\{\boldsymbol{e}_i - \boldsymbol{e}_j \,\big|\, 1 \le i < j \le k\big\} \ .$$

By setting $\langle \gamma_i, \gamma_j \rangle_{\mathfrak{Q}} := \gamma_i(H_j) = C_{ij}$, we define a nondegenerate symmetric bilinear form $\langle -,- \rangle_{\mathfrak{Q}}$ on $\mathfrak{Q}$.

The *fundamental weights* ω_i *of type* A_{k-1} are the elements of $\mathfrak{t}^*$ defined by $\omega_i(H_j) = \delta_{ij}$ for $i,j = 1,\dots,k-1$. In the standard basis of $\mathbb{Z}^k$, they are given explicitly by

$$\omega_i := \sum_{l=1}^{i} \boldsymbol{e}_l - \frac{i}{k} \sum_{l=1}^{k} \boldsymbol{e}_l$$

for $i = 1,\dots,k-1$. Let $\mathfrak{P} := \bigoplus_{i=1}^{k-1} \mathbb{Z}\omega_i$ be the *weight lattice*. Then $\mathfrak{Q} \subset \mathfrak{P}$, as $\gamma_i = \sum_{j=1}^{k-1} C_{ij}\,\omega_j$. The set of *dominant weights* is $\mathfrak{P}_+ := \bigoplus_{i=1}^{k-1} \mathbb{N}\omega_i$. There is a coset decomposition of $\mathfrak{P}$ given by

$$\mathfrak{P} = \bigcup_{j=0}^{k-1} (\mathfrak{Q} + \omega_j) \ , \tag{3.17}$$

where we set $\omega_0 := \boldsymbol{0}$.

The *coroot lattice* is the lattice $\mathfrak{Q}^\vee := \bigoplus_{i=1}^{k-1} \mathbb{Z}H_i$. $\triangle$

We now introduce the Kac-Moody algebra $\widehat{\mathfrak{sl}}_k$ of type $\widehat{A}_{k-1}$, first via its canonical generators and then as a central extension of the loop algebra of $\mathfrak{sl}_k$.

Definition 3.18. The *Kac-Moody algebra* $\widehat{\mathfrak{sl}}_k$ *of type* $\widehat{A}_{k-1}$ over $\mathbb{F}$ is the Lie algebra over $\mathbb{F}$ generated by e_i, f_i, h_i for $i = 0,1,\dots,k-1$ satisfying the relations

$$\begin{aligned} [e_i, f_j] &= \delta_{ij}\, h_j \ , \quad [h_i, h_j] = 0 \ , \\ [h_i, e_j] &= \widehat{C}_{ij}\, e_j \ , \quad [h_i, f_j] = -\widehat{C}_{ij}\, f_j \ , \end{aligned}$$

where $\widehat{C} = \big(\widehat{C}_{ij}\big)$ is the Cartan matrix of the extended Dynkin diagram of type $\widehat{A}_{k-1}$. $\oslash$

The matrix $\widehat{C}$ is given for $k \geq 3$ by

$$\widehat{C} = \big(\widehat{C}_{ij}\big) = \begin{pmatrix} 2 & -1 & 0 & \dots & -1 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 2 \end{pmatrix}$$

and for $k = 2$ by

$$\widehat{C} = \big(\widehat{C}_{ij}\big) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} .$$

Let us denote by $\widehat{\mathfrak{t}}$ the Lie subalgebra of $\widehat{\mathfrak{sl}}_k$ generated by h_i for $i = 0, 1, \dots, k-1$ and by $\widehat{\mathfrak{n}}_+$ (resp. $\widehat{\mathfrak{n}}_-$) the Lie subalgebra of $\widehat{\mathfrak{sl}}_k$ generated by e_i (resp. f_i) for $i = 0, 1, \dots, k-1$. Then there is a triangular decomposition

$$\widehat{\mathfrak{sl}}_k = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{t}} \oplus \widehat{\mathfrak{n}}_+$$

as a direct sum of vector spaces.

Now we describe the relation between $\mathfrak{sl}_k$ and $\widehat{\mathfrak{sl}}_k$. Define in $\mathfrak{sl}_k$ the elements

$$E_0 := \boldsymbol{E}_{k,1} \ , \qquad F_0 := \boldsymbol{E}_{1,k} \qquad \text{and} \qquad H_0 := \boldsymbol{E}_{k,k} - \boldsymbol{E}_{1,1} \ .$$

Consider next the *loop algebra* $\widetilde{\mathfrak{sl}}_k := \mathfrak{sl}_k \otimes \mathbb{F}[t, t^{-1}]$. Set

$$\begin{aligned} \tilde{e}_0 &:= E_0 \otimes t \ , & \tilde{e}_i &:= E_i \otimes 1 \ , \\ \tilde{f}_0 &:= F_0 \otimes t^{-1} \ , & \tilde{f}_i &:= F_i \otimes 1 \ , \\ \tilde{h}_0 &:= H_0 \otimes 1 \ , & \tilde{h}_i &:= H_i \otimes 1 \ , \end{aligned}$$

for $i = 1, \dots, k-1$. Let us denote by $\mathfrak{c}$ the central element of $\widehat{\mathfrak{sl}}_k$ given by $\mathfrak{c} = \sum_{i=0}^{k-1} h_i$. Then we can realize $\widehat{\mathfrak{sl}}_k$ as a one-dimensional central extension

$$0 \longrightarrow \mathbb{F}\mathfrak{c} \longrightarrow \widehat{\mathfrak{sl}}_k \xrightarrow{\ \pi\ } \widetilde{\mathfrak{sl}}_k \longrightarrow 0 \ ,$$

where the homomorphism π is defined by

$$\pi \, : \, e_i \longmapsto \tilde{e}_i \ , \qquad f_i \longmapsto \tilde{f}_i \ , \qquad h_i \longmapsto \tilde{h}_i \ ,$$

for $i = 0, 1, \dots, k-1$, and the Lie algebra structure of $\widehat{\mathfrak{sl}}_k$ is obtained through

$$[M \otimes t^m, N \otimes t^n] = [M, N] \otimes t^{m+n} + m \, \delta_{m,-n} \operatorname{tr}(M \, N) \, \mathfrak{c} \tag{3.19}$$

for every $M, N \in \mathfrak{sl}_k$ and $m, n \in \mathbb{Z}$. Thus the canonical generators of $\widehat{\mathfrak{sl}}_k$ are

$$\begin{aligned} e_0 &:= E_0 \otimes t \ , & e_i &:= E_i \otimes 1 \ , \\ f_0 &:= F_0 \otimes t^{-1} \ , & f_i &:= F_i \otimes 1 \ , \\ h_0 &:= H_0 \otimes 1 + \mathfrak{c} \ , & h_i &:= H_i \otimes 1 \ , \end{aligned}$$

and we can realize $\widehat{\mathfrak{t}}$ as the one-dimensional extension

$$0 \longrightarrow \mathbb{F}\mathfrak{c} \longrightarrow \widehat{\mathfrak{t}} \xrightarrow{\ \pi\ } \mathfrak{t} \longrightarrow 0 \ .$$

Remark 3.20. Let $\gamma_0 := -\sum_{i=1}^{k-1}\gamma_i$. For $i = 1,\dots,k-1$, let $\boldsymbol{e}_i$ be as in Remark 3.16; then $\gamma_0 = \boldsymbol{e}_k - \boldsymbol{e}_1$. We extend $\boldsymbol{e}_i$ from $\mathfrak{t}^*$ to $\widehat{\mathfrak{t}}^*$ by setting $\boldsymbol{e}_i(\mathfrak{c}) = 0$. Then $\gamma_i(\mathfrak{c}) = 0$ for $i = 0,1,\dots,k-1$. Thus the root lattice $\widehat{\mathfrak{Q}}$ of $\widehat{\mathfrak{sl}}_k$ is the lattice $\widehat{\mathfrak{Q}} = \bigoplus_{i=0}^{k-1}\mathbb{Z}\gamma_i = \mathbb{Z}\gamma_0 \oplus \mathfrak{Q}$. In a similar way, one can define the lattice of positive roots and a nondegenerate symmetric bilinear form on $\widehat{\mathfrak{Q}}$.

Let $\widehat{\omega}_0$ be the element in $\widehat{\mathfrak{t}}^*$ defined by $\widehat{\omega}_0(\mathfrak{t}^*) = 0$ and $\widehat{\omega}_0(\mathfrak{c}) = 1$. Define

$$\widehat{\omega}_i := \omega_i + \widehat{\omega}_0 \qquad \text{for} \quad i = 1,\dots,k-1\ .$$

We call $\widehat{\omega}_0, \widehat{\omega}_1, \dots, \widehat{\omega}_{k-1}$ the *fundamental weights of type* $\widehat{A}_{k-1}$. Set $\widehat{\mathfrak{P}} := \bigoplus_{i=0}^{k-1}\mathbb{Z}\widehat{\omega}_i$. Any weight $\widehat{\lambda} = \sum_{i=0}^{k-1}\lambda_i\,\widehat{\omega}_i \in \widehat{\mathfrak{P}}$ can be written as $\widehat{\lambda} = \lambda + k_{\widehat{\lambda}}\,\widehat{\omega}_0$, where $\lambda \in \mathfrak{P}$ and $k_{\widehat{\lambda}} = \widehat{\lambda}(\mathfrak{c}) = \sum_{i=0}^{k-1}\lambda_i$ is the *level* of $\widehat{\lambda}$. △

3.2.1 Highest weight representations

By declaring the degrees of generators $\deg e_i = -\deg f_i = 1$ and $\deg h_i = 0$ for $i = 0,1,\dots,k-1$, we endow $\widehat{\mathfrak{sl}}_k$ with the *principal grading*

$$\widehat{\mathfrak{sl}}_k = \bigoplus_{n\in\mathbb{Z}} \big(\widehat{\mathfrak{sl}}_k\big)_n\ .$$

The principal grading of $\widehat{\mathfrak{sl}}_k$ induces a $\mathbb{Z}$-grading of its universal enveloping algebra $\mathcal{U}\big(\widehat{\mathfrak{sl}}_k\big)$ over $\mathbb{F}$, which is written as

$$\mathcal{U}\big(\widehat{\mathfrak{sl}}_k\big) = \bigoplus_{n\in\mathbb{Z}} \mathcal{U}_n\ .$$

Set $\widehat{\mathfrak{b}} := \widehat{\mathfrak{t}} \oplus \widehat{\mathfrak{n}}_+$. Let $\widehat{\lambda}$ be a linear form on $\widehat{\mathfrak{t}}$. We define a one-dimensional $\widehat{\mathfrak{b}}$-module $\mathbb{F}v_{\widehat{\lambda}}$ by

$$\widehat{\mathfrak{n}}_+ \triangleright v_{\widehat{\lambda}} = 0 \qquad \text{and} \qquad h_i \triangleright v_{\widehat{\lambda}} = \widehat{\lambda}(h_i)\, v_{\widehat{\lambda}} \qquad \text{for} \quad i = 0,1,\dots,k-1\ .$$

Consider the induced $\widehat{\mathfrak{sl}}_k$-module

$$\tilde{\mathcal{V}}(\widehat{\lambda}) := \mathcal{U}\big(\widehat{\mathfrak{sl}}_k\big) \otimes_{\mathcal{U}(\widehat{\mathfrak{b}})} \mathbb{F}v_{\widehat{\lambda}}\ .$$

Setting $\tilde{\mathcal{V}}_n := \mathcal{U}_n \triangleright v_{\widehat{\lambda}}$, we define the principal grading $\tilde{\mathcal{V}}(\widehat{\lambda}) = \bigoplus_{n\in\mathbb{Z}}\tilde{\mathcal{V}}_n$. The $\widehat{\mathfrak{sl}}_k$-module $\tilde{\mathcal{V}}(\widehat{\lambda})$ contains a unique maximal proper graded $\widehat{\mathfrak{sl}}_k$-submodule $I(\widehat{\lambda})$.

Definition 3.21. The quotient module

$$\mathcal{V}(\widehat{\lambda}) := \tilde{\mathcal{V}}(\widehat{\lambda})\,/\,I(\widehat{\lambda})$$

is called the *highest weight representation of* $\widehat{\mathfrak{sl}}_k$ *at level* $k_{\widehat{\lambda}}$. The nonzero multiples of the image of $v_{\widehat{\lambda}}$ in $\mathcal{V}(\widehat{\lambda})$ are called the *highest weight vectors* of $\mathcal{V}(\widehat{\lambda})$. ⊘

The principal grading on $\tilde{\mathcal{V}}(\widehat{\lambda})$ induces an $\mathbb{N}$-grading

$$\mathcal{V}(\widehat{\lambda}) = \bigoplus_{n\geq 0} \mathcal{V}_{-n}$$

called the principal grading of $\mathcal{V}(\widehat{\lambda})$.

Definition 3.22. The *i-th dominant representation of $\widehat{\mathfrak{sl}}_k$ at level one* is the highest weight representation $\mathcal{V}(\widehat{\omega}_i)$ of $\widehat{\mathfrak{sl}}_k$ for $i = 0, 1, \dots, k-1$. The module $\mathcal{V}(\widehat{\omega}_0)$ is also called the *basic representation of $\widehat{\mathfrak{sl}}_k$*.

$\oslash$

Remark 3.23. One can define the Lie algebra $\widehat{\mathfrak{gl}}_k$ as the one-dimensional extension

$$0 \longrightarrow \mathbb{F}\mathfrak{c} \longrightarrow \widehat{\mathfrak{gl}}_k \xrightarrow{\pi} \mathfrak{gl}_k \otimes \mathbb{F}[t, t^{-1}] \longrightarrow 0 .$$

Since $\mathfrak{gl}_k = \mathbb{F}\,\mathrm{id} \oplus \mathfrak{sl}_k$, the representation theory of $\widehat{\mathfrak{gl}}_k$ is obtained by combining the representation theory of the Heisenberg algebra $\mathfrak{h}_{\mathbb{F}}$ with that of $\widehat{\mathfrak{sl}}_k$. For example, all highest weight representations of $\widehat{\mathfrak{gl}}_k$ are of the form $\mathcal{F}_{\mathbb{F}} \otimes \mathcal{V}(\widehat{\lambda})$ for some weight $\widehat{\lambda} \in \widehat{\mathfrak{P}}$. $\triangle$

3.2.2 Whittaker vectors

Let us denote by $\mathfrak{q}^i_m$ the element $H_i \otimes t^m$ for $i \in \{1, \dots, k-1\}$ and $m \in \mathbb{Z}$. By Equation (3.19), these elements satisfy

$$\big[\mathfrak{q}^i_m, \mathfrak{q}^j_n\big] = m\,\delta_{m+n,0}\,C_{ij}\,\mathfrak{c} \qquad \text{and} \qquad \big[\mathfrak{q}_m, \mathfrak{c}\big] = 0 ,$$

for $i, j \in \{1, \dots, k-1\}$ and $m, n \in \mathbb{Z}$. For a root γ, we denote by $\mathfrak{q}^\gamma_m$ the element $H_\gamma \otimes t^m$ where $H_\gamma \in \mathfrak{t}$ is defined by $\langle H, H_\gamma \rangle_{\mathfrak{Q}^\vee \otimes_{\mathbb{Z}} \mathbb{R}} = \gamma(H)$ for any $H \in \mathfrak{t}$.

The subalgebra of $\widehat{\mathfrak{sl}}_k$ generated by $\mathfrak{q}^i_m$, for $i \in \{1, \dots, k-1\}$ and $m \in \mathbb{Z} \setminus \{0\}$, and $\mathfrak{c}$ is isomorphic to the Heisenberg algebra $\mathfrak{h}_{\mathbb{F},\mathfrak{Q}}$. This motivates the following definition of Whittaker vector for $\widehat{\mathfrak{sl}}_k$ (cf. [19, Section 6]).

Definition 3.24. Let $\chi \colon \mathcal{U}(\mathfrak{h}^+_{\mathbb{F},\mathfrak{Q}}) \to \mathbb{F}$ be an algebra homomorphism such that $\chi|_{\mathfrak{h}^+_{\mathbb{F},\mathfrak{Q}}} \neq 0$, and let V be a $\mathcal{U}\big(\widehat{\mathfrak{sl}}_k\big)$-module. A nonzero vector $w \in V$ is called a *Whittaker vector of type χ* if $\eta \triangleright w = \chi(\eta)\, w$ for all $\eta \in \mathcal{U}(\mathfrak{h}^+_{\mathbb{F},\mathfrak{Q}})$. $\oslash$

3.3 Frenkel-Kac construction

Let $\mathcal{V}$ be a representation of $\mathfrak{h}_{\mathbb{F},\mathfrak{Q}}$. We say that it is a *level one* representation if the central element $\mathfrak{c}$ acts by the identity map. Henceforth we let $\mathcal{V}$ be a level one representation of $\mathfrak{h}_{\mathbb{F},\mathfrak{Q}}$ such that for any $v \in \mathcal{V}$ there exists an integer $m(v)$ for which

$$\big(\mathfrak{q}^{l_1}_{m_1} \cdots \mathfrak{q}^{l_a}_{m_a}\big) \triangleright v = 0 \tag{3.25}$$

if $m_i > 0$ and $\sum_i m_i > m(v)$.

Fix an index $j \in \{0, 1, \dots, k-1\}$ and consider the coset $\mathfrak{Q} + \omega_j$. Denote by $\mathbb{F}[\mathfrak{Q} + \omega_j]$ the group algebra of $\mathfrak{Q} + \omega_j$ over $\mathbb{F}$. For a root $\gamma \in \mathfrak{Q}$ we define the generating function $V(\gamma, z) \in \mathrm{End}(\mathcal{V} \otimes \mathbb{F}[\mathfrak{Q} + \omega_j])[[z, z^{-1}]]$ of operators on $\mathcal{V} \otimes \mathbb{F}[\mathfrak{Q} + \omega_j]$ by the bosonic vertex operator

$$\begin{aligned} V(\gamma, z) &= \mathrm{V}^{\gamma}_{1,-1}(z)\ \exp(\log z\ \mathfrak{c} + \gamma) \\ &= \exp\Big(\sum_{m=1}^{\infty} \frac{z^m}{m}\,\mathfrak{q}^\gamma_{-m}\Big)\ \exp\Big(-\sum_{m=1}^{\infty} \frac{z^{-m}}{m}\,\mathfrak{q}^\gamma_{m}\Big)\ \exp(\log z\ \mathfrak{c} + \gamma) , \end{aligned}$$

where $\exp(\log z\, \mathfrak{c}+\gamma)$ is the operator defined by

$$\exp(\log z\, \mathfrak{c}+\gamma) \triangleright (v\otimes[\beta+\omega_j]) := z^{\frac{1}{2}\langle\gamma,\gamma\rangle_{\mathfrak{Q}}+\langle\gamma,\beta+\omega_j\rangle_{\mathfrak{Q}}}\,(v\otimes[\beta+\gamma+\omega_j])$$

for $v\otimes[\beta+\omega_j]\in\mathcal{V}\otimes\mathbb{F}[\mathfrak{Q}+\omega_j]$.

Remark 3.26. Here for the operator $\exp(\log z\, \mathfrak{c}+\gamma)$ we follow the notation in [42, Section 3.2.1]. In the existing literature, this operator is denoted in various different ways. △

Let $V_m(\gamma)\in\mathrm{End}(\mathcal{V}\otimes\mathbb{F}[\mathfrak{Q}+\omega_j])$ denote the operator defined by the formal Laurent series expansion $V(\gamma,z)=\sum_{m\in\mathbb{Z}}V_m(\gamma)\,z^m$.

We define a map $\epsilon\colon\mathfrak{Q}\times\mathfrak{Q}\to\{\pm1\}$ by

$$\epsilon(\gamma_i,\gamma_j)=\begin{cases}-1\,, & j=i,i+1\,,\\ 1\,, & \text{otherwise}\,,\end{cases}$$

with the properties $\epsilon(\gamma+\gamma',\beta)=\epsilon(\gamma,\beta)\,\epsilon(\gamma',\beta)$ and $\epsilon(\gamma,\beta+\beta')=\epsilon(\gamma,\beta)\,\epsilon(\gamma,\beta')$.

Theorem 3.27 ([26, Theorem 1])**.** *Let $j\in\{0,1,\dots,k-1\}$ and let $\mathcal{V}$ be a level one representation of $\mathfrak{h}_{\mathbb{F},\mathfrak{Q}}$ satisfying the condition* (3.25)*. Then the vector space $\mathcal{V}\otimes\mathbb{F}[\mathfrak{Q}+\omega_j]$ carries a level one $\widehat{\mathfrak{sl}}_k$-module structure given by*

$$\begin{aligned}(H_i\otimes 1)\triangleright(v\otimes[\beta+\omega_j]) &= \big(\langle\gamma_i,\beta\rangle_{\mathfrak{Q}}+\delta_{ij}\big)\,(v\otimes[\beta+\omega_j])\,,\\ (H_i\otimes t^m)\triangleright(v\otimes[\beta+\omega_j]) &= \big(\mathfrak{q}^i_m\triangleright v\big)\otimes[\beta+\omega_j]\,,\\ (E_i\otimes t^m)\triangleright(v\otimes[\beta+\omega_j]) &= \epsilon(\gamma_i,\beta)\,V_{m+\delta_{ij}}(\gamma_i)\triangleright(v\otimes[\beta+\omega_j])\,,\\ (F_i\otimes t^m)\triangleright(v\otimes[\beta+\omega_j]) &= \epsilon(\beta,\gamma_i)\,V_{-m-\delta_{ij}}(-\gamma_i)\triangleright(v\otimes[\beta+\omega_j])\,,\end{aligned}$$

for $i\in\{1,\dots,k-1\}$ and $m\in\mathbb{Z}\setminus\{0\}$. If $\mathcal{V}$ is the Fock space of $\mathfrak{h}_{\mathbb{F},\mathfrak{Q}}$, then $\mathcal{V}\otimes\mathbb{F}[\mathfrak{Q}+\omega_j]$ is the j-th dominant representation of $\widehat{\mathfrak{sl}}_k$.

3.3.1 Virasoro operators

Let $\{\eta_i\}_{i=1}^{k-1}$ be an orthonormal basis of the vector space $\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{R}$. The Virasoro algebra associated with $\mathfrak{h}_{\mathbb{F},\mathfrak{Q}}\subset\widehat{\mathfrak{sl}}_k$ has generators $\mathfrak{c}$ and $L_n^{\widehat{\mathfrak{sl}}_k}$ for $n\in\mathbb{Z}$ defined by [26, Section 2.8]

$$\begin{aligned}L_0^{\widehat{\mathfrak{sl}}_k} &= \sum_{i=1}^{k-1}\sum_{m=1}^{\infty}\mathfrak{q}^{\eta_i}_{-m}\,\mathfrak{q}^{\eta_i}_{m}+\frac{1}{2}\sum_{i=1}^{k-1}\big(\mathfrak{q}^{\eta_i}_0\big)^2\,,\\ L_n^{\widehat{\mathfrak{sl}}_k} &= \frac{1}{2}\sum_{i=1}^{k-1}\sum_{m\in\mathbb{Z}}\mathfrak{q}^{\eta_i}_{-m}\,\mathfrak{q}^{\eta_i}_{m+n}\qquad\text{for}\quad n\in\mathbb{Z}\setminus\{0\}\,.\end{aligned}$$

Note that distinct orthonormal bases of $\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{R}$ give rise to the same Virasoro algebra $\mathfrak{Vir}_{\mathbb{F}}$.

4 AGT relations on $\mathbb{R}^4$

4.1 Equivariant cohomology of $\mathrm{Hilb}^n(\mathbb{C}^2)$

In the following we shall give a brief survey of results concerning the equivariant cohomology of the Hilbert schemes $\mathrm{Hilb}^n(\mathbb{C}^2)$ and representations of Heisenberg algebras thereon [43, 29, 47, 60, 36, 55, 44].

Let us consider the action of the torus $T := (\mathbb{C}^*)^2$ on the complex affine plane $\mathbb{C}^2$ given by $(t_1, t_2) \triangleright (x, y) = (t_1\, x, t_2\, y)$, and the induced T-action on the Hilbert scheme of n points $\mathrm{Hilb}^n(\mathbb{C}^2)$ which is the fine moduli space parameterizing zero-dimensional subschemes of $\mathbb{C}^2$ of length n; it is a smooth quasi-projective variety of dimension $2n$. Following [22], the T-fixed points of $\mathrm{Hilb}^n(\mathbb{C}^2)$ are zero-dimensional subschemes of $\mathbb{C}^2$ of length n supported at the origin $0 \in \mathbb{C}^2$ which correspond to partitions λ of n. We shall denote by Z_λ the fixed point in $\mathrm{Hilb}^n(\mathbb{C}^2)^T$ corresponding to the partition λ of n.

For $i = 1, 2$ denote by t_i the T-modules corresponding to the characters $\chi_i \colon (t_1, t_2) \in T \mapsto t_i \in \mathbb{C}^*$, and by ε_i the equivariant first Chern class of t_i. Then $H^*_T(\mathrm{pt}; \mathbb{C}) = \mathbb{C}[\varepsilon_1, \varepsilon_2]$ is the coefficient ring for the T-equivariant cohomology. The equivariant Chern character of the tangent space to $\mathrm{Hilb}^n(\mathbb{C}^2)$ at a fixed point Z_λ is given by

$$\mathrm{ch}_T\big(T_{Z_\lambda}\mathrm{Hilb}^n(\mathbb{C}^2)\big) = \sum_{s\in Y_\lambda} \big(\, \mathrm{e}^{\,(L(s)+1)\,\varepsilon_1 - A(s)\,\varepsilon_2} + \mathrm{e}^{\,-L(s)\,\varepsilon_1 + (A(s)+1)\,\varepsilon_2}\big) \ .$$

The equivariant Euler class is therefore given by

$$\mathrm{eu}_T\,\big(T_{Z_\lambda}\mathrm{Hilb}^n(\mathbb{C}^2)\big) = (-1)^n\ \mathrm{eu}_+(\lambda)\ \mathrm{eu}_-(\lambda)\ ,$$

where

$$\mathrm{eu}_+(\lambda) = \prod_{s\in Y_\lambda} \big((L(s)+1)\,\varepsilon_1 - A(s)\,\varepsilon_2\big) \qquad \text{and} \qquad \mathrm{eu}_-(\lambda) = \prod_{s\in Y_\lambda} \big(L(s)\,\varepsilon_1 - (A(s)+1)\,\varepsilon_2\big)\ .$$

Remark 4.1. By [44, Corollary 3.20], $\mathrm{eu}_+(\lambda)$ is the equivariant Euler class of the nonpositive part $T^{\leq 0}_{Z_\lambda}$ of the tangent space to $\mathrm{Hilb}^n(\mathbb{C}^2)$ at the fixed point Z_λ. △

Let $\imath_\lambda \colon \{Z_\lambda\} \hookrightarrow \mathrm{Hilb}^n(\mathbb{C}^2)$ be the inclusion morphism and define the class

$$[\lambda] := \imath_{\lambda *}(1) \ \in\ H^{4n}_T\big(\mathrm{Hilb}^n(\mathbb{C}^2)\big)\ . \tag{4.2}$$

By the projection formula we get

$$[\lambda] \cup [\mu] = \delta_{\lambda,\mu}\ \mathrm{eu}_T\,\big(T_{Z_\lambda}\mathrm{Hilb}^n(\mathbb{C}^2)\big)\,[\lambda] = (-1)^n\,\delta_{\lambda,\mu}\ \mathrm{eu}_+(\lambda)\ \mathrm{eu}_-(\lambda)\,[\lambda]\ .$$

Denote

$$\imath_n := \bigoplus_{Z_\lambda\in\mathrm{Hilb}^n(\mathbb{C}^2)^T} \imath_\lambda :\ \mathrm{Hilb}^n(\mathbb{C}^2)^T \ \longrightarrow\ \mathrm{Hilb}^n(\mathbb{C}^2)\ .$$

Let $\imath_n^! \colon H^*_T\,\big(\mathrm{Hilb}^n(\mathbb{C}^2)^T\big)_{\mathrm{loc}} \to H^*_T\big(\mathrm{Hilb}^n(\mathbb{C}^2)\big)_{\mathrm{loc}}$ be the induced Gysin map, where

$$H^*_T(-)_{\mathrm{loc}} := H^*_T(-) \otimes_{\mathbb{C}[\varepsilon_1,\varepsilon_2]} \mathbb{C}(\varepsilon_1, \varepsilon_2)$$

is the localized equivariant cohomology. By the localization theorem, $\imath_n^!$ is an isomorphism and its inverse is given by

$$\left(\imath_n^!\right)^{-1} \,:\, A \longmapsto \Big(\frac{\imath_\lambda^*(A)}{\mathrm{eu}_T\left(T_{Z_\lambda}\mathrm{Hilb}^n(\mathbb{C}^2)\right)}\Big)_{Z_\lambda\in\mathrm{Hilb}^n(\mathbb{C}^2)^T} .$$

Henceforth we denote $\mathbb{H}_{\mathbb{C}^2,n} := H_T^*(\mathrm{Hilb}^n(\mathbb{C}^2))_{\mathrm{loc}}$. Define the bilinear form

$$\begin{aligned} \langle -,-\rangle_{\mathbb{H}_{\mathbb{C}^2,n}} \,:\, \mathbb{H}_{\mathbb{C}^2,n}\times\mathbb{H}_{\mathbb{C}^2,n} &\longrightarrow \mathbb{C}(\varepsilon_1,\varepsilon_2) , \\ (A,B) &\longmapsto (-1)^n\, p_n^!\left(\imath_n^!\right)^{-1}(A\cup B) , \end{aligned} \tag{4.3}$$

where p_n is the projection of $\mathrm{Hilb}^n(\mathbb{C}^2)^T$ to a point.

Remark 4.4. Our sign convention in defining the bilinear form is different from the one used e.g. in [47, 18]. We choose this convention because, under the isomorphism to be introduced later on in (4.12), the form (4.3) becomes exactly the Jack inner product (2.1). This convention produces various sign changes compared to previous literature. Hence every time we state that a given result coincides with what is known in the literature, the reader should keep in mind "up to the sign convention we choose". △

Following [36, Section 2.2] we define the distinguished classes

$$[\alpha_\lambda] := \frac{1}{\mathrm{eu}_+(\lambda)}\,[\lambda] \,\in\, H_T^{2n}\big(\mathrm{Hilb}^n(\mathbb{C}^2)\big)_{\mathrm{loc}} .$$

For λ,μ partitions of n one has

$$\begin{aligned} \big\langle[\alpha_\lambda]\,,\,[\alpha_\mu]\big\rangle_{\mathbb{H}_{\mathbb{C}^2,n}} &= \delta_{\lambda,\mu}\,\frac{\mathrm{eu}_-(\lambda)}{\mathrm{eu}_+(\lambda)} \\ &= \delta_{\lambda,\mu}\prod_{s\in Y_\lambda}\frac{L(s)\,\varepsilon_1-\big(A(s)+1\big)\,\varepsilon_2}{\big(L(s)+1\big)\,\varepsilon_1-A(s)\varepsilon_2} = \delta_{\lambda,\mu}\prod_{s\in Y_\lambda}\frac{L(s)\,\beta+A(s)+1}{\big(L(s)+1\big)\,\beta+A(s)} , \end{aligned} \tag{4.5}$$

where

$$\beta = -\frac{\varepsilon_1}{\varepsilon_2} . \tag{4.6}$$

Remark 4.7. In [44, Section 3(v)], Nakajima gives a geometric interpretation of the class $[\alpha_\lambda]$. △

By the localization theorem and Equation (4.5), the classes $[\alpha_\lambda]$ form a $\mathbb{C}(\varepsilon_1,\varepsilon_2)$-basis for the infinite-dimensional vector space $\mathbb{H}_{\mathbb{C}^2} := \bigoplus_{n\geq 0}\mathbb{H}_{\mathbb{C}^2,n}$. Hence the symmetric bilinear form (4.3) is nondegenerate. The forms $\langle -,-\rangle_{\mathbb{H}_{\mathbb{C}^2,n}}$ define a symmetric bilinear form

$$\langle -,-\rangle_{\mathbb{H}_{\mathbb{C}^2}} \,:\, \mathbb{H}_{\mathbb{C}^2}\times\mathbb{H}_{\mathbb{C}^2} \longrightarrow \mathbb{C}(\varepsilon_1,\varepsilon_2)$$

by imposing that $\mathbb{H}_{\mathbb{C}^2,n_1}$ and $\mathbb{H}_{\mathbb{C}^2,n_2}$ are orthogonal for $n_1\neq n_2$. Then $\langle -,-\rangle_{\mathbb{H}_{\mathbb{C}^2}}$ is also nondegenerate.

The unique partition of $n=1$ is $\lambda=(1)$. Let us denote by $[\alpha] := [\alpha_{(1)}]$ the corresponding class. Then

$$\big\langle[\alpha]\,,\,[\alpha]\big\rangle_{\mathbb{H}_{\mathbb{C}^2}} = \beta^{-1} .$$

Let us denote by D_x and D_y respectively the x and y axes of $\mathbb{C}^2$. By localization, the corresponding equivariant cohomology classes in $H_T^*(\mathbb{C}^2)_{\mathrm{loc}}$ are given by

$$[D_x]_T = \frac{[0]}{\varepsilon_1} = \frac{[0]}{\mathrm{eu}_+(1)} = [\alpha] \qquad \text{and} \qquad [D_y]_T = \frac{[0]}{\varepsilon_2} = -\beta\,[\alpha] .$$

4.2 Heisenberg algebra

Following [43, 47], for an integer $m > 0$ define the Hecke correspondences

$$D_x(n,m) := \big\{ (Z, Z') \in \mathrm{Hilb}^{n+m}(\mathbb{C}^2) \times \mathrm{Hilb}^n(\mathbb{C}^2) \,\big|\, Z' \subset Z \ , \ \mathrm{supp}(\mathcal{I}_{Z'}/\mathcal{I}_Z) = \{y\} \subset D_x \big\} \ ,$$

where $\mathcal{I}_Z, \mathcal{I}_{Z'}$ are the ideal sheaves corresponding to Z, Z' respectively. Let q_1, q_2 denote the projections of $\mathrm{Hilb}^{n+m}(\mathbb{C}^2) \times \mathrm{Hilb}^n(\mathbb{C}^2)$ to the two factors, respectively. Define linear operators $\mathfrak{p}_{-m}([D_x]_T) \in \mathrm{End}(\mathbb{H}_{\mathbb{C}^2})$ by

$$\mathfrak{p}_{-m}([D_x]_T)A := q_1^! \big(q_2^* A \cup [D_x(n,m)]_T \big)$$

for $A \in H_T^*(\mathrm{Hilb}^n(\mathbb{C}^2))_{\mathrm{loc}}$. We also define $\mathfrak{p}_m([D_x]_T) \in \mathrm{End}(\mathbb{H}_{\mathbb{C}^2})$ to be the adjoint operator of $\mathfrak{p}_{-m}([D_x]_T)$ with respect to the inner product $\langle -, - \rangle_{\mathbb{H}_{\mathbb{C}^2}}$ on $\mathbb{H}_{\mathbb{C}^2}$. As the class $[D_x]_T$ spans $H_T^*(\mathbb{C}^2)_{\mathrm{loc}}$ over the field $\mathbb{C}(\varepsilon_1, \varepsilon_2)$, we can define operators $\mathfrak{p}_m(\eta) \in \mathrm{End}(\mathbb{H}_{\mathbb{C}^2})$ for every class $\eta \in H_T^*(\mathbb{C}^2)_{\mathrm{loc}}$.

Theorem 4.8 (see [43, 44])**.** *The linear operators $\mathfrak{p}_m(\eta)$, for $m \in \mathbb{Z} \setminus \{0\}$ and $\eta \in H_T^*(\mathbb{C}^2)_{\mathrm{loc}}$, satisfy the Heisenberg commutation relations*

$$\big[\mathfrak{p}_m(\eta_1) \, , \, \mathfrak{p}_n(\eta_2) \big] = m \, \delta_{m,-n} \, \langle \eta_1, \eta_2 \rangle_{\mathbb{H}_{\mathbb{C}^2,1}} \, \mathrm{id} \qquad \textit{and} \qquad \big[\mathfrak{p}_m(\eta) \, , \, \mathrm{id} \big] = 0 \ .$$

The vector space $\mathbb{H}_{\mathbb{C}^2}$ becomes the Fock space of the Heisenberg algebra $\mathfrak{h}_{\mathbb{H}_{\mathbb{C}^2,1}}$ modelled on $\mathbb{H}_{\mathbb{C}^2,1} = H_T^*(\mathbb{C}^2)_{\mathrm{loc}}$ with the unit $|0\rangle$ in $H_T^0(\mathrm{Hilb}^0(\mathbb{C}^2))_{\mathrm{loc}}$ as highest weight vector.

Remark 4.9. Since $[D_x]_T = [\alpha]$, we have $\mathfrak{p}_m([\alpha]) = \mathfrak{p}_m([D_x]_T)$. △

Henceforth we denote by $\mathfrak{h}_{\mathbb{C}^2}$ the Heisenberg algebra $\mathfrak{h}_{\mathbb{H}_{\mathbb{C}^2,1}}$, and we define

$$\mathfrak{p}_m := \mathfrak{p}_m([D_x]_T) \qquad \text{for} \quad m \in \mathbb{Z} \setminus \{0\} \ , \tag{4.10}$$

so that one has the nonzero commutation relations

$$[\mathfrak{p}_{-m}, \mathfrak{p}_m] = m \, \beta^{-1} \, \mathrm{id} \ .$$

Since $[D_x]_T$ generates $H_T^*(\mathbb{C}^2)_{\mathrm{loc}}$ over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$, the operators $\mathfrak{p}_m$ generate $\mathfrak{h}_{\mathbb{C}^2}$.

Let $\lambda = (1^{m_1} \, 2^{m_2} \, \cdots)$ be a partition. Define $\mathfrak{p}_\lambda := \prod_i \mathfrak{p}_{-i}^{m_i}$. Then

$$\big\langle \mathfrak{p}_\lambda |0\rangle \, , \, \mathfrak{p}_\mu |0\rangle \big\rangle_{\mathbb{H}_{\mathbb{C}^2}} = \delta_{\lambda,\mu} \, z_\lambda \, \beta^{-\ell(\lambda)} \ .$$

Let us denote by Λ_β the ring of symmetric functions in infinitely many variables $\Lambda_{\mathbb{C}(\varepsilon_1,\varepsilon_2)}$ over the field $\mathbb{C}(\varepsilon_1, \varepsilon_2)$, equiped with the Jack inner product (2.1).

Theorem 4.11 (see [43, 36, 18])**.** *There exists a $\mathbb{C}(\varepsilon_1, \varepsilon_2)$-linear isomorphism*

$$\phi : \, \mathbb{H}_{\mathbb{C}^2} \ \longrightarrow \ \Lambda_\beta \tag{4.12}$$

preserving bilinear forms such that

$$\phi(\mathfrak{p}_\lambda |0\rangle) = p_\lambda(x) \qquad \textit{and} \qquad \phi([\alpha_\lambda]) = J_\lambda(x; \beta^{-1}) \ .$$

Via the isomorphism ϕ, the operators $\mathfrak{p}_m$ act on Λ_β as multiplication by p_{-m} for $m < 0$ and as $m \, \beta^{-1} \, \frac{\partial}{\partial p_m}$ for $m > 0$.

4.2.1 Whittaker vectors

We characterize a particular class of Whittaker vectors (cf. Definition 3.14) which will be useful in our studies of gauge theories.

Proposition 4.13. *Let $\eta \in \mathbb{C}(\varepsilon_1, \varepsilon_2)$. In the completed Fock space $\prod_{n\geq 0} \mathbb{H}_{\mathbb{C}^2,n}$, every vector of the form*

$$G(\eta) := \exp\left(\eta\, \mathfrak{p}_{-1}\right) |0\rangle$$

is a Whittaker vector of type χ_η, where the algebra homomorphism $\chi_\eta \colon \mathcal{U}(\mathfrak{h}^+_{\mathbb{C}^2}) \to \mathbb{C}(\varepsilon_1, \varepsilon_2)$ is defined by

$$\chi_\eta(\mathfrak{p}_1) = \eta\, \beta^{-1} \qquad \textit{and} \qquad \chi_\eta(\mathfrak{p}_n) = 0 \qquad \textit{for} \quad n > 1 \ .$$

Proof. The statement follows from the formal expansion

$$G(\eta) = \sum_{n=0}^{\infty} \frac{\eta^n}{n!}\, (\mathfrak{p}_{-1})^n |0\rangle \tag{4.14}$$

with respect to the vector $|0\rangle$, together with the relation $\mathfrak{p}_m|0\rangle = 0$ for $m > 0$ and the identity

$$\mathfrak{p}_m\, (\mathfrak{p}_{-1})^n = n\, \beta^{-1}\, \delta_{m,1}\, (\mathfrak{p}_{-1})^{n-1} + (\mathfrak{p}_{-1})^n\, \mathfrak{p}_m$$

in $\mathcal{U}(\mathfrak{h}_{\mathbb{C}^2})$ for $m \geq 1$. □

4.3 Vertex operators

Let $T_\mu = \mathbb{C}^*$ and $H^*_{T_\mu}(\mathrm{pt};\mathbb{C}) = \mathbb{C}[\mu]$. Let us denote by $\mathcal{O}_{\mathbb{C}^2}(\mu)$ the trivial line bundle on $\mathbb{C}^2$ on which T_μ acts by scaling the fibers. In [18], Carlsson and Okounkov define a vertex operator $\mathrm{V}(\mathcal{L}, z)$ for any smooth quasi-projective surface X and any line bundle $\mathcal{L}$ on X. Here we shall describe only $\mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu), z)$; see [18] for a complete description of such types of vertex operators.

Let $\boldsymbol{Z}_n \subset \mathrm{Hilb}^n(\mathbb{C}^2) \times \mathbb{C}^2$ be the universal subscheme, whose fiber over a point $Z \in \mathrm{Hilb}^n(\mathbb{C}^2)$ is the subscheme $Z \subset \mathbb{C}^2$ itself. Consider

$$\boldsymbol{\mathcal{Z}}_i := p^*_{i3}(\mathcal{O}_{\boldsymbol{Z}_{n_i}}) \ \in \ K\big(\mathrm{Hilb}^{n_1}(\mathbb{C}^2) \times \mathrm{Hilb}^{n_2}(\mathbb{C}^2) \times \mathbb{C}^2\big) \qquad \text{for} \quad i = 1, 2 \ ,$$

where p_{ij} denotes the projection to the i-th and j-th factors. Define the virtual vector bundle

$$\boldsymbol{E}^{n_1,n_2}_\mu = p_{12*}\left((\boldsymbol{\mathcal{Z}}_1^\vee + \boldsymbol{\mathcal{Z}}_2 - \boldsymbol{\mathcal{Z}}_1^\vee \cdot \boldsymbol{\mathcal{Z}}_2) \cdot p_3^*(\mathcal{O}_{\mathbb{C}^2}(\mu))\right) \ \in \ K\big(\mathrm{Hilb}^{n_1}(\mathbb{C}^2) \times \mathrm{Hilb}^{n_2}(\mathbb{C}^2)\big) \ ,$$

where p_3 is the projection to $\mathbb{C}^2$. The fibre of $\boldsymbol{E}^{n_1,n_2}_\mu$ over $(Z_1, Z_2) \in \mathrm{Hilb}^{n_1}(\mathbb{C}^2)^T \times \mathrm{Hilb}^{n_2}(\mathbb{C}^2)^T$ is given by

$$\boldsymbol{E}^{n_1,n_2}_\mu\big|_{(Z_1,Z_2)} = \chi\big(\mathcal{O}_{\mathbb{C}^2}\, ,\, \mathcal{O}_{\mathbb{C}^2}(\mu)\big) - \chi\big(\mathcal{I}_{Z_1}\, ,\, \mathcal{I}_{Z_2} \otimes \mathcal{O}_{\mathbb{C}^2}(\mu)\big) \ ,$$

where $\chi(E, F) := \sum_{i=0}^2 (-1)^i\, \mathrm{Ext}^i(E, F)$ for any pair of coherent sheaves E, F on $\mathbb{C}^2$, while its rank is

$$\mathrm{rk}\left(\boldsymbol{E}^{n_1,n_2}_\mu\right) = n_1 + n_2 \ .$$

Define the operator $\mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu), z) \in \mathrm{End}(\mathbb{H}_{\mathbb{C}^2})[[z, z^{-1}]]$ by its matrix elements

$$(-1)^{n_2} \left\langle \mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu), z) A_1\, ,\, A_2 \right\rangle_{\mathbb{H}_{\mathbb{C}^2}}$$

$$:= z^{n_2-n_1} \int_{\mathrm{Hilb}^{n_1}(\mathbb{C}^2)\times \mathrm{Hilb}^{n_2}(\mathbb{C}^2)} \mathrm{eu}_T\left(\boldsymbol{E}_\mu^{n_1,n_2}\right)\cup p_1^*(A_1)\cup p_2^*(A_2)\ , \tag{4.15}$$

where $A_i \in H_T^*(\mathrm{Hilb}^{n_i}(\mathbb{C}^2))_{\mathrm{loc}}$ and p_i is the projection from $\mathrm{Hilb}^{n_1}(\mathbb{C}^2)\times \mathrm{Hilb}^{n_2}(\mathbb{C}^2)$ to the i-th factor for $i=1,2$. By [18, Lemma 6], the matrix elements (4.15) in the fixed point basis are given by

$$\begin{aligned}\left\langle \mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu),z)[\lambda_1]\,,\,[\lambda_2]\right\rangle_{\mathbb{H}_{\mathbb{C}^2}} &= (-1)^{|\lambda_2|}\, z^{|\lambda_2|-|\lambda_1|}\ \mathrm{eu}_T\left(\boldsymbol{E}_\mu^{n_1,n_2}\big|_{(Z_{\lambda_1},Z_{\lambda_2})}\right) \\ &= (-1)^{|\lambda_2|}\, z^{|\lambda_2|-|\lambda_1|}\ m_{Y_{\lambda_1},Y_{\lambda_2}}(\varepsilon_1,\varepsilon_2,\mu)\ ,\end{aligned} \tag{4.16}$$

where

$$\begin{aligned} m_{Y_1,Y_2}(\varepsilon_1,\varepsilon_2,a) := \prod_{s_1\in Y_1} &\left(a - L_{Y_2}(s_1)\,\varepsilon_1 + (A_{Y_1}(s_1)+1)\,\varepsilon_2\right) \\ \times \prod_{s_2\in Y_2} &\left(a + (L_{Y_1}(s_2)+1)\,\varepsilon_1 - A_{Y_2}(s_2)\,\varepsilon_2\right)\end{aligned} \tag{4.17}$$

for a pair of Young tableaux Y_1, Y_2 and $a\in\mathbb{C}(\varepsilon_1,\varepsilon_2)$. In gauge theory this factorized expression for the matrix elements represents the contribution of the *bifundamental hypermultiplet*.

We shall now describe the operator $\mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu),z)$ in terms of the operators $\mathfrak{p}_m$ defined in Equation (4.10) for $m\in\mathbb{Z}\setminus\{0\}$. In our setting, [18, Theorem 1] assumes the following form.

Theorem 4.18. *The operator* $\mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu),z)$ *is a vertex operator in Heisenberg operators given by the generalized bosonic exponential associated with the Heisenberg algebra* $\mathfrak{h}_{\mathbb{C}^2}$ *as*

$$\mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu),z) = \mathrm{V}_{-\frac{\mu}{\varepsilon_2},\,\frac{\mu+\varepsilon_1+\varepsilon_2}{\varepsilon_2}}(z)\ . \tag{4.19}$$

4.4 Integrals of motion

Let $\boldsymbol{V}^n$ be the pushforward of $\boldsymbol{E}_0^{n,n}$ with respect to the projection of the product $\mathrm{Hilb}^n(\mathbb{C}^2)\times \mathrm{Hilb}^n(\mathbb{C}^2)$ to the second factor. It is a T-equivariant vector bundle on $\mathrm{Hilb}^n(\mathbb{C}^2)$ of rank n, which we shall call the natural bundle over $\mathrm{Hilb}^n(\mathbb{C}^2)$. The T-equivariant Chern character of $\boldsymbol{V}^n$ at the fixed point Z_λ is given by

$$\mathrm{ch}_T\left(\boldsymbol{V}^n\big|_{Z_\lambda}\right) = \sum_{s\in Y_\lambda} \mathrm{e}^{-L'(s)\,\varepsilon_1 - A'(s)\,\varepsilon_2}\ .$$

Remark 4.20. The vector bundle $\boldsymbol{V}^n$ can equivalently be defined as the pushforward with respect to the projection $\mathrm{Hilb}^n(\mathbb{C}^2)\times\mathbb{C}^2\to \mathrm{Hilb}^n(\mathbb{C}^2)$ of the structure sheaf of the universal subscheme $\boldsymbol{Z}_n$. In the literature $\boldsymbol{V}^n$ is also called the tautological sheaf and denoted $\mathcal{O}^{[n]}$ (cf. [34, 35, 54, 44]). △

Let us denote by $\boldsymbol{V}$ the natural bundle over $\coprod_{n\geq 0}\mathrm{Hilb}^n(\mathbb{C}^2)$. The operators of multiplication by $\boldsymbol{I}_1 := \mathrm{rk}\left(\boldsymbol{V}\right)$ and $\boldsymbol{I}_p := (\mathrm{c}_{p-1})_T\left(\boldsymbol{V}\right)$ for $p\geq 2$ on $\prod_{n\geq 0}\mathbb{H}_{\mathbb{C}^2,n}$ have even degrees, are self-adjoint with respect to the inner product on $\prod_{n\geq 0}\mathbb{H}_{\mathbb{C}^2,n}$, and commute with each other; they can thus be simultaneously diagonalized in the fixed point basis $[\lambda]$ of $\prod_{n\geq 0}\mathbb{H}_{\mathbb{C}^2,n}$ (see [54], and also [44, Section 4] where our operator $\mathfrak{p}_m$ is denoted $P_m(\varepsilon_2)$ for $m\in\mathbb{Z}\setminus\{0\}$). For example, one has

$$\boldsymbol{I}_1 \triangleright [\lambda] = |\lambda|\,[\lambda] \qquad \text{and} \qquad \boldsymbol{I}_2\triangleright[\lambda] = -\sum_{s\in Y_\lambda}\left(L'(s)\,\varepsilon_1 + A'(s)\,\varepsilon_2\right)[\lambda]\ . \tag{4.21}$$

As a consequence, the operators of multiplication by $\boldsymbol{I}_p$ for $p \geq 1$ can be written in terms of the Heisenberg operators (4.10) as elements of a commutative subalgebra of $\mathcal{U}(\mathfrak{h}_{\mathbb{C}^2})$. For example, we have

$$\boldsymbol{I}_1 = \beta \sum_{m=1}^{\infty} \mathfrak{p}_{-m}\, \mathfrak{p}_m \, ,$$

$$\boldsymbol{I}_2 = \varepsilon_1 \Big(\frac{\beta}{2} \sum_{m,n=1}^{\infty} \big(\mathfrak{p}_{-m}\, \mathfrak{p}_{-n}\, \mathfrak{p}_{m+n} + \mathfrak{p}_{-m-n}\, \mathfrak{p}_n\, \mathfrak{p}_m \big) - \frac{\beta-1}{2} \sum_{m=1}^{\infty} (m-1)\, \mathfrak{p}_{-m}\, \mathfrak{p}_m \Big) \, .$$

Note that the *energy operator* $\boldsymbol{I}_1$ coincides with the Virasoro generator $L_0^{\mathfrak{h}}$ from Section 3.1.1, while the operator $\boldsymbol{I}_2$ is equal to $\varepsilon_1\, \square^{\beta^{-1}}$, where $\square^{\beta^{-1}}$ is the bosonized Hamiltonian of the quantum trigonometric Calogero-Sutherland model with infinitely many particles and coupling constant β^{-1}.

4.5 $\mathcal{N} = 2$ gauge theory

The Nekrasov partition function for pure $\mathcal{N} = 2$ $U(1)$ gauge theory on $\mathbb{R}^4$ is given by the generating function [50, 13]

$$\begin{aligned} \mathcal{Z}_{\mathbb{C}^2}(\varepsilon_1, \varepsilon_2; \mathsf{q}) &:= \sum_{n=0}^{\infty} \mathsf{q}^n \int_{\mathrm{Hilb}^n(\mathbb{C}^2)} \big[\mathrm{Hilb}^n(\mathbb{C}^2)\big]_T \\ &= \sum_{n=0}^{\infty} (-\mathsf{q})^n \big\langle [\mathrm{Hilb}^n(\mathbb{C}^2)]_T \, , \, [\mathrm{Hilb}^n(\mathbb{C}^2)]_T \big\rangle_{\mathbb{H}_{\mathbb{C}^2}} \end{aligned}$$

where $\mathsf{q} \in \mathbb{C}^*$ with $|\mathsf{q}| < 1$. By the localization theorem we obtain

$$\mathcal{Z}_{\mathbb{C}^2}(\varepsilon_1, \varepsilon_2; \mathsf{q}) = \sum_{\lambda} \Big(-\frac{\mathsf{q}}{\varepsilon_2^2} \Big)^{|\lambda|} \prod_{s \in Y_\lambda} \frac{1}{\big((L(s)+1)\,\beta + A(s)\big)\, \big(L(s)\,\beta + (A(s)+1)\big)}$$

as in [13, Equation (3.16)], where the sum runs over all partitions λ.

Remark 4.22. By [48, Equation (4.7)], the partition function can be summed explicitly and written in the closed form

$$\mathcal{Z}_{\mathbb{C}^2}(\varepsilon_1, \varepsilon_2; \mathsf{q}) = \exp\Big(\frac{\mathsf{q}}{\varepsilon_1\, \varepsilon_2} \Big) \, . \tag{4.23}$$

△

4.5.1 Gaiotto state

In [27], Gaiotto considers the inducing state of the (completed) Verma module of the Virasoro algebra. It has the property that it is a Whittaker vector for the Verma module, and the norm of its q-deformation coincides with the Nekrasov partition function of pure $\mathcal{N} = 2$ $SU(2)$ gauge theory on $\mathbb{R}^4$. Here we consider the analogous vector for $U(1)$ gauge theory on $\mathbb{R}^4$.

Following [56], we define the *Gaiotto state* to be the sum of all fundamental classes

$$G := \sum_{n \geq 0} \big[\mathrm{Hilb}^n(\mathbb{C}^2)\big]_T$$

in the completed Fock space $\prod_{n\geq 0} \mathbb{H}_{\mathbb{C}^2,n}$. We also introduce the *weighted Gaiotto state* as the formal power series

$$G_{\mathsf{q}} := \sum_{n\geq 0} \mathsf{q}^n \, \big[\mathrm{Hilb}^n(\mathbb{C}^2)\big]_T \ \in \ \prod_{n\geq 0} \mathsf{q}^n \, \mathbb{H}_{\mathbb{C}^2,n} \ .$$

Consider the bilinear form

$$\langle -,-\rangle_{\mathbb{H}_{\mathbb{C}^2},\mathsf{q}} \ : \ \prod_{n\geq 0} \mathsf{q}^n \, \mathbb{H}_{\mathbb{C}^2,n} \times \prod_{n\geq 0} \mathsf{q}^n \, \mathbb{H}_{\mathbb{C}^2,n} \ \longrightarrow \ \mathbb{C}(\varepsilon_1,\varepsilon_2)[[\mathsf{q}]]$$

defined by

$$\Big\langle \sum_{n\geq 0} \mathsf{q}^n \, \eta_n \, , \, \sum_{n\geq 0} \mathsf{q}^n \, \nu_n \Big\rangle_{\mathbb{H}_{\mathbb{C}^2},\mathsf{q}} := \sum_{n=0}^\infty \mathsf{q}^n \, \int_{\mathrm{Hilb}^n(\mathbb{C}^2)} \eta_n \cup \nu_n = \sum_{n=0}^\infty (-\mathsf{q})^n \, \langle \eta_n,\nu_n\rangle_{\mathbb{H}_{\mathbb{C}^2}} \ .$$

It follows immediately that the norm of the weighted Gaiotto state is the Nekrasov partition function for $\mathcal{N}=2$ $U(1)$ gauge theory on $\mathbb{R}^4$:

$$\mathcal{Z}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2;\mathsf{q}) = \langle G_{\mathsf{q}}, G_{\mathsf{q}}\rangle_{\mathbb{H}_{\mathbb{C}^2},\mathsf{q}} \ .$$

By Proposition 4.13 we have the following result.

Proposition 4.24. *The Gaiotto state G is a Whittaker vector of type χ, where the algebra homomorphism $\chi\colon \mathcal{U}(\mathfrak{h}^+_{\mathbb{C}^2}) \to \mathbb{C}(\varepsilon_1,\varepsilon_2)$ is defined by*

$$\chi(\mathfrak{p}_1) = -\frac{1}{\varepsilon_1} \qquad \textit{and} \qquad \chi(\mathfrak{p}_n) = 0 \qquad \textit{for} \quad n>1 \ .$$

Proof. Let $\eta \in \mathbb{C}(\varepsilon_1,\varepsilon_2)$. By using the formal expansion (4.14) and the isomorphism ϕ, we can write $\phi(G(\eta))$ in terms of powers p_1^n. By Lemma 2.3 and simple algebraic manipulations we can then rewrite the vector $G(\eta)$ as

$$G(\eta) = \sum_{n\geq 0} (\eta\,\varepsilon_2)^n \, \big[\mathrm{Hilb}^n(\mathbb{C}^2)\big]_T$$

and the result follows. □

4.6 Quiver gauge theories

We now add matter fields to the $\mathcal{N}=2$ gauge theory on $\mathbb{R}^4$. We consider the most general $\mathcal{N}=2$ superconformal quiver gauge theory with gauge group $U(1)^{r+1}$ for $r\geq 0$, following the general ADE classification of [52, Chapter 3].

Let $\mathtt{Q}=(\mathtt{Q}_0,\mathtt{Q}_1)$ be a quiver, i.e., an oriented graph with a finite set of vertices $\mathtt{Q}_0$, a finite set of edges $\mathtt{Q}_1\subset \mathtt{Q}_0\times\mathtt{Q}_0$, and two projection maps $\mathtt{s},\mathtt{t} : \mathtt{Q}_1 \rightrightarrows \mathtt{Q}_0$ which assign to each oriented edge its source and target vertex respectively. Representations of the quiver encode the matter field content of the gauge theory. Fix a vector $(n_\upsilon)_{\upsilon\in\mathtt{Q}_0}\in\mathbb{N}^{\mathtt{Q}_0}$ of integers labelled by the nodes of the quiver $\mathtt{Q}$, and consider the product of Hilbert schemes $\prod_{\upsilon\in\mathtt{Q}_0}\mathrm{Hilb}^{n_\upsilon}(\mathbb{C}^2)$. The vertices $\upsilon\in\mathtt{Q}_0$ label $U(1)$ gauge groups and $m_\upsilon\geq 0$ (resp. $\bar m_\upsilon \geq 0$) fundamental (resp. antifundamental) hypermultiplets of masses μ_υ^s, $s=1,\dots,m_\upsilon$ (resp. $\bar\mu_\upsilon^{\bar s}$, $\bar s = 1,\dots,\bar m_\upsilon$) which correspond to the T-equivariant vector bundles $\boldsymbol{V}^{n_\upsilon}_{\mu_\upsilon^s}$ (resp. $\bar{\boldsymbol{V}}^{n_\upsilon}_{\bar\mu_\upsilon^{\bar s}}$) of rank n_υ on $\mathrm{Hilb}^{n_\upsilon}(\mathbb{C}^2)$ obtained by pushforward of $\boldsymbol{E}^{n_\upsilon,n_\upsilon}_{\mu_\upsilon^s}$ (resp. $\boldsymbol{E}^{n_\upsilon,n_\upsilon}_{\bar\mu_\upsilon^{\bar s}}$) with respect to the projection of $\mathrm{Hilb}^{n_\upsilon}(\mathbb{C}^2)\times\mathrm{Hilb}^{n_\upsilon}(\mathbb{C}^2)$ to the second (resp. first) factor. The edges $e \in \mathtt{Q}_1$ label $U(1) \times U(1)$ bifundamental hypermultiplets of masses μ_e which correspond to the vector bundles $\boldsymbol{E}_{\mu_e}^{n_{\mathtt{s}(e)},n_{\mathtt{t}(e)}}$ of rank $n_{\mathtt{s}(e)} + n_{\mathtt{t}(e)}$ on $\mathrm{Hilb}^{n_{\mathtt{s}(e)}}(\mathbb{C}^2) \times \mathrm{Hilb}^{n_{\mathtt{t}(e)}}(\mathbb{C}^2)$; if the edge e is a vertex loop, i.e., $\mathtt{s}(e) = \mathtt{t}(e)$, then the restriction of $\boldsymbol{E}_{\mu_e}^{n_{\mathtt{s}(e)},n_{\mathtt{s}(e)}}$ to the diagonal of $\mathrm{Hilb}^{n_{\mathtt{s}(e)}}(\mathbb{C}^2) \times \mathrm{Hilb}^{n_{\mathtt{s}(e)}}(\mathbb{C}^2)$ describes an adjoint hypermultiplet of mass μ_e. The total matter field content of the $\mathcal{N} = 2$ quiver gauge theory associated to $\mathtt{Q}$ in the sector labelled by $(n_\upsilon)_{\upsilon\in\mathtt{Q}_0} \in \mathbb{N}^{\mathtt{Q}_0}$ is thus described by the bundle on $\prod_{\upsilon\in\mathtt{Q}_0} \mathrm{Hilb}^{n_\upsilon}(\mathbb{C}^2)$ given by

$$\boldsymbol{M}^{(n_\upsilon)}_{(\mu_\upsilon^s),(\bar{\mu}_\upsilon^{\bar{s}}),(\mu_e)} := \bigoplus_{\upsilon\in\mathtt{Q}_0} p_\upsilon^*\Big(\bigoplus_{s=1}^{m_\upsilon} \boldsymbol{V}^{n_\upsilon}_{\mu_\upsilon^s} \oplus \bigoplus_{\bar{s}=1}^{\bar{m}_\upsilon} \bar{\boldsymbol{V}}^{n_\upsilon}_{\bar{\mu}_\upsilon^{\bar{s}}} \Big) \oplus \bigoplus_{e\in\mathtt{Q}_1} p_e^* \boldsymbol{E}_{\mu_e}^{n_{\mathtt{s}(e)},n_{\mathtt{t}(e)}} ,$$

where p_υ is the projection of $\prod_{\upsilon\in\mathtt{Q}_0} \mathrm{Hilb}^{n_\upsilon}(\mathbb{C}^2)$ to the υ-th factor and p_e the projection to $\mathrm{Hilb}^{n_{\mathtt{s}(e)}}(\mathbb{C}^2)\times \mathrm{Hilb}^{n_{\mathtt{t}(e)}}(\mathbb{C}^2)$.

For each vertex $\upsilon \in \mathtt{Q}_0$, the degree of the Euler class of the pushforward of this bundle to the υ-th factor is the integer

$$\begin{aligned} d_\upsilon &:= \dim_{\mathbb{C}} \mathrm{Hilb}^{n_\upsilon}(\mathbb{C}^2) - \mathrm{rk}\,\Big(\boldsymbol{M}^{(n_\upsilon)}_{(\mu_\upsilon^s),(\bar{\mu}_\upsilon^{\bar{s}}),(\mu_e)}\Big|_{\mathrm{Hilb}^{n_\upsilon}(\mathbb{C}^2)} \Big) \\ &= n_\upsilon \big(2 - m_\upsilon - \bar{m}_\upsilon - \#\{ e \in \mathtt{Q}_1 \,|\, \mathtt{s}(e) = \upsilon \} - \#\{ e \in \mathtt{Q}_1 \,|\, \mathtt{t}(e) = \upsilon \} \big) . \end{aligned}$$

The $\mathcal{N} = 2$ quiver gauge theory is said to be *conformal* if $d_\upsilon = 0$ for all $\upsilon \in \mathtt{Q}_0$; it is *asymptotically free* if $d_\upsilon > 0$. Note that with this definition the pure $\mathcal{N} = 2$ gauge theory of Section 4.5 is asymptotically free. As explained in [52, Chapter 3], $\mathcal{N} = 2$ asymptotically free quiver gauge theories can be recovered from conformal theories, so in the following we restrict our attention to superconformal quiver gauge theories.

Introduce coupling constants $\mathsf{q}_\upsilon \in \mathbb{C}^*$ with $|\mathsf{q}_\upsilon| < 1$ at each vertex $\upsilon \in \mathtt{Q}_0$, and let $T_{\boldsymbol{\mu}}$ be the maximal torus of the total flavour symmetry group

$$G_{\mathrm{f}} = \prod_{\upsilon\in\mathtt{Q}_0} GL(m_\upsilon, \mathbb{C}) \times GL(\bar{m}_\upsilon, \mathbb{C}) \times \prod_{e\in\mathtt{Q}_1} \mathbb{C}^*$$

with $H^*_{T_{\boldsymbol{\mu}}}(\mathrm{pt}; \mathbb{C}) = \mathbb{C}[(\mu_e), (\mu_\upsilon^s), (\bar{\mu}_\upsilon^{\bar{s}})]$. Then the quiver gauge theory partition function is defined by the generating function

$$\begin{aligned} \mathcal{Z}^{\mathtt{Q}}_{\mathbb{C}^2}(\varepsilon_1, \varepsilon_2, \boldsymbol{\mu}; \mathbf{q}) &:= \sum_{(n_\upsilon)\in\mathbb{N}^{\mathtt{Q}_0}} \mathbf{q}^{\boldsymbol{n}} \int_{\prod\limits_{\upsilon\in\mathtt{Q}_0} \mathrm{Hilb}^{n_\upsilon}(\mathbb{C}^2)} \mathrm{eu}_{T\times T_{\boldsymbol{\mu}}} \big(\boldsymbol{M}^{(n_\upsilon)}_{(\mu_\upsilon^s),(\bar{\mu}_\upsilon^{\bar{s}}),(\mu_e)} \big) \\ &= \sum_{(n_\upsilon)\in\mathbb{N}^{\mathtt{Q}_0}} \mathbf{q}^{\boldsymbol{n}} \int_{\prod\limits_{\upsilon\in\mathtt{Q}_0} \mathrm{Hilb}^{n_\upsilon}(\mathbb{C}^2)} \prod_{\upsilon\in\mathtt{Q}_0} p_\upsilon^* \Big(\prod_{s=1}^{m_\upsilon} \mathrm{eu}_T \big(\boldsymbol{V}^{n_\upsilon}_{\mu_\upsilon^s} \big) \prod_{\bar{s}=1}^{\bar{m}_\upsilon} \mathrm{eu}_T \big(\bar{\boldsymbol{V}}^{n_\upsilon}_{\bar{\mu}_\upsilon^{\bar{s}}} \big) \Big) \\ &\qquad \times \prod_{e\in\mathtt{Q}_1} p_e^* \, \mathrm{eu}_T \big(\boldsymbol{E}_{\mu_e}^{n_{\mathtt{s}(e)},n_{\mathtt{t}(e)}} \big) , \end{aligned}$$

where $\mathbf{q}^{\boldsymbol{n}} := \prod_{\upsilon\in\mathtt{Q}_0} \mathsf{q}_\upsilon^{n_\upsilon}$. By the localization theorem, we obtain

$$\mathcal{Z}^{\mathtt{Q}}_{\mathbb{C}^2}(\varepsilon_1, \varepsilon_2, \boldsymbol{\mu}; \mathbf{q}) = \sum_{(\lambda^\upsilon)} (-\mathbf{q})^{\boldsymbol{\lambda}} \prod_{\upsilon\in\mathtt{Q}_0} \frac{\prod\limits_{s=1}^{m_\upsilon} m_{Y_{\lambda^\upsilon}}(\varepsilon_1, \varepsilon_2, \mu_\upsilon^s) \prod\limits_{\bar{s}=1}^{\bar{m}_\upsilon} m_{Y_{\lambda^\upsilon}}(\varepsilon_1, \varepsilon_2, \bar{\mu}_\upsilon^{\bar{s}} + \varepsilon_1 + \varepsilon_2)}{m_{Y_{\lambda^\upsilon}, Y_{\lambda^\upsilon}}(\varepsilon_1, \varepsilon_2, 0)}$$

$$\times \prod_{e\in\mathtt{Q}_1} m_{Y_{\lambda^{\mathtt{s}(e)}},Y_{\lambda^{\mathtt{t}(e)}}}(\varepsilon_1,\varepsilon_2,\mu_e) \tag{4.25}$$

where $\mathsf{q}^{\boldsymbol{\lambda}} := \prod_{\upsilon\in\mathtt{Q}_0} \mathsf{q}_\upsilon^{|\lambda^\upsilon|}$ for a collection of partitions λ^υ associated to the vertices of the quiver, and

$$m_Y(\varepsilon_1,\varepsilon_2,a) := \prod_{s\in Y} \big(a - L'(s)\,\varepsilon_1 - A'(s)\,\varepsilon_2\big)$$

for a Young tableau Y and $a\in\mathbb{C}(\varepsilon_1,\varepsilon_2)$.

The conformal constraint

$$m_\upsilon + \bar m_\upsilon + \#\big\{e\in\mathtt{Q}_1\,\big|\,\mathtt{s}(e)=\upsilon\big\} + \#\big\{e\in\mathtt{Q}_1\,\big|\,\mathtt{t}(e)=\upsilon\big\} = 2 \tag{4.26}$$

for each $\upsilon\in\mathtt{Q}_0$ severely restricts the possible quivers in the abelian gauge theory. It is easy to check that the only admissible quivers in the ADE classification of [52, Chapter 3] are the linear (or chain) quivers of the finite-dimensional A_r-type Dynkin diagram and the cyclic (or necklace) quivers of the affine $\widehat{A}_r$-type extended Dynkin diagram for some $r\geq 0$[1]. We consider in detail each case in turn.

4.7 $\widehat{A}_r$ theories

For the cyclic quivers of type $\widehat{A}_r$

$$\begin{tikzcd}
 & & \circ \arrow[dll] & & \\
\circ \arrow[r] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r] & \circ \arrow[ull]
\end{tikzcd}$$

with $r+1$ vertices and arrows, one has $m_\upsilon=\bar m_\upsilon=0$ by Equation (4.26). We label the vertices $\mathtt{Q}_0$ by $\upsilon=0,1,\dots,r$ with counterclockwise orientation and read modulo $r+1$, and similarly for the edges $e=(\upsilon,\upsilon+1)\in\mathtt{Q}_1$. The partition function for the $\mathcal{N}=2$ quiver gauge theory of type $\widehat{A}_r$ reads as

$$\mathcal{Z}^{\widehat{A}_r}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathsf{q}) = \sum_{\boldsymbol{\lambda}} (-\mathsf{q})^{\boldsymbol{\lambda}} \prod_{\upsilon=0}^{r} \frac{m_{Y_{\lambda^\upsilon},Y_{\lambda^{\upsilon+1}}}(\varepsilon_1,\varepsilon_2,\mu_\upsilon)}{m_{Y_{\lambda^\upsilon},Y_{\lambda^\upsilon}}(\varepsilon_1,\varepsilon_2,0)}\ , \tag{4.27}$$

where the sum is over all $r+1$-vectors of partitions $\boldsymbol{\lambda}=(\lambda^0,\lambda^1,\dots,\lambda^r)$ with $\lambda^{r+1}:=\lambda^0$ and $\mathsf{q}^{\boldsymbol{\lambda}} := \prod_{\upsilon=0}^{r}\mathsf{q}_\upsilon^{|\lambda^\upsilon|}$.

4.7.1 Conformal blocks

We will relate the partition function (4.27) to the trace of vertex operators $\mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu_\upsilon),z_\upsilon)$. We shall also denote by $\mathfrak{h}$ the Heisenberg algebra $\mathfrak{h}_{\mathbb{C}^2}$ to simplify the presentation.

Proposition 4.28. *The partition function of the $\widehat{A}_r$-theory on $\mathbb{R}^4$ is given by*

$$\mathcal{Z}^{\widehat{A}_r}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathsf{q}) = \mathrm{Tr}_{\mathbb{H}_{\mathbb{C}^2}}\,\mathsf{q}^{L_0^{\mathfrak{h}}} \prod_{\upsilon=0}^{r} \mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu_\upsilon),z_\upsilon)$$

independently of $z_0\in\mathbb{C}^*$, where $\mathsf{q}:=\mathsf{q}_0\,\mathsf{q}_1\cdots\mathsf{q}_r$ and $z_\upsilon := z_0\,\mathsf{q}_1\cdots\mathsf{q}_\upsilon$ for $\upsilon=1,\dots,r$.

[1]Here the A_0-type Dynkin diagram is the trivial quiver consisting of a single vertex with no arrows, and the $\widehat{A}_0$-type Dynkin diagram is the quiver consisting of a single vertex with a vertex edge loop.

Proof. By Equation (4.21), the Virasoro operator $L_0^{\mathfrak{h}}$ acts in $\mathbb{H}_{\mathbb{C}^2}$ as

$$L_0^{\mathfrak{h}}\big|_{\mathbb{H}_{\mathbb{C}^2,n}} = n\ \mathrm{id}_{\mathbb{H}_{\mathbb{C}^2,n}} \ ,$$

and so the trace of products of Ext vertex operators $\mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu_\upsilon), z_\upsilon)$ is given by the sum of their matrix elements over the fixed point basis as

$$\mathrm{Tr}_{\mathbb{H}_{\mathbb{C}^2}}\, \mathsf{q}^{L_0^{\mathfrak{h}}} \prod_{\upsilon=0}^{r} \mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu_\upsilon), z_\upsilon) = \sum_{\boldsymbol{n}\in\mathbb{N}^{r+1}} \mathsf{q}^{n_0} \sum_{\boldsymbol{\lambda}\,:\,|\lambda^\upsilon|=n_\upsilon}\ \prod_{\upsilon=0}^{r} \frac{\big\langle \mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu_\upsilon), z_\upsilon)[\lambda^\upsilon]\,,\,[\lambda^{\upsilon+1}]\big\rangle_{\mathbb{H}_{\mathbb{C}^2}}}{\big\langle[\lambda^\upsilon]\,,\,[\lambda^\upsilon]\big\rangle_{\mathbb{H}_{\mathbb{C}^2}}} \ .$$

By Equation (4.16) we obtain

$$\begin{aligned}\mathrm{Tr}_{\mathbb{H}_{\mathbb{C}^2}}\, \mathsf{q}^{L_0^{\mathfrak{h}}} \prod_{\upsilon=0}^{r} \mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu_\upsilon), 1) &= \sum_{\boldsymbol{n}\in\mathbb{N}^{r+1}} \mathsf{q}^{n_0} \prod_{\upsilon=0}^{r} (-1)^{n_\upsilon}\, z_\upsilon^{n_\upsilon-n_{\upsilon+1}} \\ &\qquad\times \sum_{\boldsymbol{\lambda}\,:\,|\lambda^\upsilon|=n_\upsilon}\ \prod_{\upsilon=0}^{r} \frac{\mathrm{eu}_T\big(\boldsymbol{E}_{\mu_\upsilon}^{n_\upsilon,n_{\upsilon+1}}\big|_{(Z_{\lambda^\upsilon},Z_{\lambda^{\upsilon+1}})}\big)}{\mathrm{eu}_T\big(T_{Z_{\lambda^\upsilon}}\mathrm{Hilb}^{n_\upsilon}(\mathbb{C}^2)\big)} \\ &= \sum_{\boldsymbol{n}\in\mathbb{N}^{r+1}} (-\mathbf{q})^{\boldsymbol{n}} \sum_{\boldsymbol{\lambda}\,:\,|\lambda^\upsilon|=n_\upsilon}\ \prod_{\upsilon=0}^{r} \frac{m_{Y_{\lambda^\upsilon},Y_{\lambda^{\upsilon+1}}}(\varepsilon_1,\varepsilon_2,\mu_\upsilon)}{m_{Y_{\lambda^\upsilon},Y_{\lambda^\upsilon}}(\varepsilon_1,\varepsilon_2,0)} \ ,\end{aligned}$$

and the result follows. □

Remark 4.29. Proposition 4.28 shows that the partition function of the $\widehat{A}_r$-theory coincides with the conformal block of the Heisenberg algebra $\mathfrak{h}_{\mathbb{C}^2}$ on the elliptic curve with nome q and $r+1$ punctures at $z_0, z_1, \ldots, z_r$; we can set $z_0 = 1$ without loss of generality. The conformal dimension of the primary field inserted at the υ-th puncture is

$$\Delta(\mu_\upsilon;\varepsilon_1,\varepsilon_2) = \frac{\mu_\upsilon\,(\mu_\upsilon+\varepsilon_1+\varepsilon_2)}{2\varepsilon_1\,\varepsilon_2} \ .$$

This elliptic curve is the Seiberg-Witten curve of the $\mathcal{N}=2$ $U(1)^{r+1}$ quiver gauge theory on $\mathbb{R}^4$. △

By using the same arguments as in the proof of [18, Corollary 1], an explicit formula for the trace in this case can be obtained using Equation (4.19) and we arrive at the explicit evaluation of the partition function. In the following $\eta(\mathsf{q}) := \mathsf{q}^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-\mathsf{q}^n)$ denotes the Dedekind function.

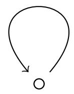
Proposition 4.30.

$$\mathcal{Z}_{\mathbb{C}^2}^{\widehat{A}_r}(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathbf{q}) = \prod_{\upsilon=0}^{r} \big(\mathsf{q}_\upsilon^{-\frac{1}{24}}\,\eta(\mathsf{q}_\upsilon)\big)^{-\frac{\mu_\upsilon\,(\mu_\upsilon+\varepsilon_1+\varepsilon_2)}{\varepsilon_1\,\varepsilon_2}}\ \mathsf{q}^{\frac{1}{24}}\,\eta(\mathsf{q})^{-1} \ .$$

A similar formula for the $U(1)^{r+1}$ quiver gauge theory partition function is conjectured in [3, Appendix C.2].

4.7.2 $\widehat{A}_0$ theory

The degenerate case $r = 0$ of the $\widehat{A}_r$ quiver gauge theory corresponds to the quiver consisting of a single node with a vertex edge loop



and is known as the $\mathcal{N} = 2^*$ gauge theory; it describes a single adjoint matter hypermultiplet of mass μ in the $U(1)$ $\mathcal{N} = 2$ gauge theory on $\mathbb{R}^4$. Then the quiver gauge theory partition function is the Nekrasov partition function for $\mathcal{N} = 2^*$ gauge theory [50, 13] and is given by

$$\mathcal{Z}_{\mathbb{C}^2}^{\widehat{A}_0}(\varepsilon_1, \varepsilon_2, \mu; \mathsf{q}) = \sum_{\lambda} \mathsf{q}^{|\lambda|} \prod_{s \in Y_\lambda} \frac{\big((L(s)+1)\,\varepsilon_1 - A(s)\,\varepsilon_2 + \mu\big)\,\big(L(s)\,\varepsilon_1 - (A(s)+1)\,\varepsilon_2 - \mu\big)}{\big((L(s)+1)\,\varepsilon_1 - A(s)\,\varepsilon_2\big)\,\big(L(s)\,\varepsilon_1 - (A(s)+1)\,\varepsilon_2\big)}$$

as in [13, Equation (3.26)].

By Proposition 4.30, we have

$$\mathcal{Z}_{\mathbb{C}^2}^{\widehat{A}_0}(\varepsilon_1, \varepsilon_2, \mu; \mathsf{q}) = \big(\mathsf{q}^{-\frac{1}{24}}\,\eta(\mathsf{q})\big)^{-\frac{\mu\,(\mu+\varepsilon_1+\varepsilon_2)}{\varepsilon_1\,\varepsilon_2}-1} . \tag{4.31}$$

A similar formula is written in [62, Equation (2.28)]. In the case of an antidiagonal torus action, i.e., $\varepsilon_1 = -\varepsilon_2$, this result coincides with the formula derived in [51, Equation (6.12)]. We can then rewrite Proposition 4.28 in the following way.

Corollary 4.32.

$$\mathcal{Z}_{\mathbb{C}^2}^{\widehat{A}_r}(\varepsilon_1, \varepsilon_2, \boldsymbol{\mu}; \mathbf{q}) = \prod_{\upsilon=0}^{r} \mathcal{Z}_{\mathbb{C}^2}^{\widehat{A}_0}(\varepsilon_1, \varepsilon_2, \mu_\upsilon; \mathsf{q}_\upsilon)\ \frac{\prod_{\upsilon=0}^{r} \eta(\mathsf{q}_\upsilon)}{\eta(\mathsf{q})} .$$

4.8 A_r theories

Consider now the linear quivers of type A_r

$$\bullet \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ \longrightarrow \bullet$$

with $r+1$ vertices and r arrows, where the solid nodes indicate the insertion of a single fundamental or antifundamental hypermultiplet. In this case we label vertices Q_0 from left to right with $\upsilon = 0, 1, \dots, r$ and edges Q_1 with $e = (\upsilon, \upsilon+1)$; for definiteness we take $\bar{m}_\upsilon = 0$, so that $m_0 = m_r = 1$ by the conformal constraints (4.26). The partition function for the $\mathcal{N} = 2$ quiver gauge theory of type A_r for $r \geq 1$ reads as

$$\mathcal{Z}_{\mathbb{C}^2}^{A_r}(\varepsilon_1, \varepsilon_2, \boldsymbol{\mu}; \mathbf{q}) = \sum_{\boldsymbol{\lambda}} (-\mathbf{q})^{\boldsymbol{\lambda}}\ \frac{m_{Y_{\lambda^0}}(\varepsilon_1, \varepsilon_2, \mu_0) \prod_{\upsilon=0}^{r-1} m_{Y_{\lambda^\upsilon}, Y_{\lambda^{\upsilon+1}}}(\varepsilon_1, \varepsilon_2, \mu_{\upsilon+1})\ m_{Y_{\lambda^r}}(\varepsilon_1, \varepsilon_2, \mu_{r+1})}{\prod_{\upsilon=0}^{r} m_{Y_{\lambda^\upsilon}, Y_{\lambda^\upsilon}}(\varepsilon_1, \varepsilon_2, 0)} . \tag{4.33}$$

4.8.1 Conformal blocks

We will express the partition function (4.33) as a particular matrix element of Ext vertex operators.

Proposition 4.34. *The partition function of the A_r-theory on $\mathbb{R}^4$ is given by*

$$\mathcal{Z}_{\mathbb{C}^2}^{A_r}(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathbf{q}) = \Big\langle |0\rangle\, ,\, \prod_{\upsilon=0}^{r+1} \mathrm{V}\big(\mathcal{O}_{\mathbb{C}^2}(\mu_\upsilon), z_\upsilon\big)|0\rangle \Big\rangle_{\mathbb{H}_{\mathbb{C}^2}}$$

independently of $z_0 \in \mathbb{C}^*$, where $z_\upsilon := z_0\, \mathsf{q}_0\, \mathsf{q}_1 \cdots \mathsf{q}_{\upsilon-1}$ for $\upsilon = 1, \dots, r+1$.

Proof. Arguing as in the proof of Proposition 4.28, we write

$$\begin{aligned}
&\Big\langle |0\rangle\, ,\, \prod_{\upsilon=0}^{r+1} \mathrm{V}\big(\mathcal{O}_{\mathbb{C}^2}(\mu_\upsilon), z_\upsilon\big)|0\rangle \Big\rangle_{\mathbb{H}_{\mathbb{C}^2}} \\
&= \sum_{\boldsymbol{n}\in\mathbb{N}^{r+1}} \ \sum_{\boldsymbol{\lambda}\,:\,|\lambda^\upsilon|=n_\upsilon} \big\langle \mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu_0), z_0)\,[\lambda^0],\, |0\rangle \big\rangle_{\mathbb{H}_{\mathbb{C}^2}} \frac{\prod_{\upsilon=0}^{r-1} \big\langle \mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu_{\upsilon+1}), z_{\upsilon+1})[\lambda^{\upsilon+1}]\, ,\, [\lambda^\upsilon] \big\rangle_{\mathbb{H}_{\mathbb{C}^2}}}{\prod_{\upsilon=0}^{r} \big\langle [\lambda^\upsilon]\, ,\, [\lambda^\upsilon] \big\rangle_{\mathbb{H}_{\mathbb{C}^2}}} \\
&\qquad \times \big\langle \mathrm{V}(\mathcal{O}_{\mathbb{C}^2}(\mu_{r+1}), z_{r+1})|0\rangle\, ,\, [\lambda^r] \big\rangle_{\mathbb{H}_{\mathbb{C}^2}}\ ,
\end{aligned}$$

and by Equation (4.16) and the orthogonality relation (4.5) the result then follows. □

Remark 4.35. Proposition 4.34 expresses the partition function of the A_r-theory as a conformal block of the Heisenberg algebra $\mathfrak{h}_{\mathbb{C}^2}$ on the Riemann sphere with $r+4$ punctures at $\infty, z_0, z_1, \dots, z_{r+1}, 0$; again we can set $z_0 = 1$ without loss of generality. The conformal dimension of the primary field at the insertion point z_υ is $\Delta(\mu_\upsilon;\varepsilon_1,\varepsilon_2)$, while at $\infty, 0$ they are given respectively by $\Delta(\tilde{\mu}_{\infty,0};\varepsilon_1,\varepsilon_2)$, where the masses $\tilde{\mu}_{\infty,0}$ obey

$$\tilde{\mu}_\infty + \tilde{\mu}_0 = \varepsilon_1 + \varepsilon_2 + \sum_{\upsilon=0}^{r+1} \mu_\upsilon\ .$$

The Seiberg-Witten curve of the $\mathcal{N}=2$ $U(1)^{r+1}$ quiver gauge theory on $\mathbb{R}^4$ is a branched cover of this $(r+2)$-punctured Riemann sphere, ramified over the points $\infty, 0$. △

Using the vertex operator representation, we can again get a closed formula for the combinatorial expansion (4.33).

Proposition 4.36.

$$\mathcal{Z}_{\mathbb{C}^2}^{A_r}(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathbf{q}) = \prod_{0\le \upsilon<\upsilon'\le r+1} \big(1 - \mathsf{q}_{\upsilon+1}\cdots \mathsf{q}_{\upsilon'}\big)^{-\frac{\mu_{\upsilon'}\,(\mu_\upsilon+\varepsilon_1+\varepsilon_2)}{\varepsilon_1\,\varepsilon_2}}\ .$$

Proof. Using Equation (3.5) to express the product of vertex operators in Proposition 4.34 in normal ordered form, we can write

$$\prod_{\upsilon=0}^{r+1} \mathrm{V}\big(\mathcal{O}_{\mathbb{C}^2}(\mu_\upsilon), z_\upsilon\big)|0\rangle = \prod_{0\le \upsilon<\upsilon'\le r+1} \Big(1 - \frac{z_{\upsilon'}}{z_\upsilon}\Big)^{-\frac{\mu_{\upsilon'}\,(\mu_\upsilon+\varepsilon_1+\varepsilon_2)}{\varepsilon_1\,\varepsilon_2}}$$

$$\times \exp\Big(-\sum_{v=0}^{r+1}\frac{\mu_v}{\varepsilon_2}\sum_{m=1}^{\infty}\frac{z_v^m}{m}\,\mathfrak{p}_{-m}\Big)|0\rangle$$

since $\mathfrak{p}_m|0\rangle = 0$ for all $m > 0$. Since $\mathfrak{p}_m$ is the adjoint operator of $\mathfrak{p}_{-m}$ with respect to the scalar product on $\mathbb{H}_{\mathbb{C}^2}$, we have $\big\langle|0\rangle, (\mathfrak{p}_{-m})^n|0\rangle\big\rangle_{\mathbb{H}_{\mathbb{C}^2}} = 0$ for all $m, n \geq 1$ and the result follows. □

A similar formula for the $U(1)^{r+1}$ quiver gauge theory partition function is conjectured in [3, Appendix C.1].

4.8.2 A_0 theory

The degenerate limit $r = 0$ of the A_r quiver gauge theory is built on the trivial quiver consisting of a single vertex with no arrows

$$\bullet$$

and $m_0 = 2$ fundamental matter fields by Equation (4.26). Then the quiver gauge theory partition function is the Nekrasov partition function for $\mathcal{N} = 2$ gauge theory with two fundamental matter hypermultiplets of masses μ_0, μ_1 [50, 13] which is given by

$$\mathcal{Z}^{A_0}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\mu_0,\mu_1;\mathsf{q}) = \sum_{\lambda}(-\mathsf{q})^{|\lambda|}\prod_{s\in Y_\lambda}\frac{\big(L'(s)\,\beta - A'(s)+\tilde{\mu}_0\big)\big(L'(s)\,\beta - A'(s)+\tilde{\mu}_1\big)}{\big((L(s)+1)\,\beta + A(s)\big)\big(L(s)\,\beta + A(s)+1\big)}$$

as in [13, Equation (3.22)], where $\tilde{\mu}_0 = \mu_0/\varepsilon_2$ and $\tilde{\mu}_1 = \mu_1/\varepsilon_2$. By Proposition 4.34 this partition function computes the four-point conformal block for the Heisenberg algebra $\mathfrak{h}_{\mathbb{C}^2}$ on the Riemann sphere with primary field insertions at $\infty, 1, \mathsf{q}, 0$, and by Proposition 4.36 the combinatorial sum can be evaluated explicitly with the result

$$\mathcal{Z}^{A_0}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\mu_0,\mu_1;\mathsf{q}) = (1-\mathsf{q})^{-\frac{\mu_1\,(\mu_0+\varepsilon_1+\varepsilon_2)}{\varepsilon_1\,\varepsilon_2}}\ . \tag{4.37}$$

A similar expression is written in [62, Equation (2.27)]. In the antidiagonal limit $\beta = 1$, this formula coincides with the partition function expression derived in [38, Equation (49)].

5 Moduli spaces of framed sheaves

5.1 Orbifold compactification of X_k

In this subsection we recall the construction of the orbifold compactification of the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_k$ from [14, Section 3] and describe the main results that we will use in this paper. For background to the theory of root and toric stacks used in the construction, see [14, Section 2], and to the theory of framed sheaves on (projective) Deligne-Mumford stacks, see [15].

Fix an integer $k \geq 2$ and denote by μ_k the group of k-th roots of unity in $\mathbb{C}$. A choice of a primitive k-th root of unity ω defines an isomorphism of groups $\mu_k \simeq \mathbb{Z}_k$. We define an action of $\mu_k \simeq \mathbb{Z}_k$ on $\mathbb{C}^2$ as $\omega \triangleright (x, y) := (\omega\, x, \omega^{-1}\, y)$. The quotient $\mathbb{C}^2/\mathbb{Z}_k$ is a normal affine toric surface. The origin is the only singular point of $\mathbb{C}^2/\mathbb{Z}_k$, and is a particular case of a rational double point or du Val singularity [21, Definition 10.4.10].

Let $\varphi_k\colon X_k \to \mathbb{C}^2/\mathbb{Z}_k$ be the minimal resolution of the singularity of $\mathbb{C}^2/\mathbb{Z}_k$; it is a smooth toric surface with k torus-fixed points $p_1,\dots,p_k$ and $k+1$ torus-invariant divisors $D_0, D_1,\dots,D_k$ which are smooth projective curves of genus zero. For any $i=1,\dots,k$ the divisors D_{i-1} and D_i intersect at the point p_i. Moreover, $D_1,\dots,D_{k-1}$ are the irreducible components of the exceptional divisor $\varphi_k^{-1}(0)$. By the McKay correspondence, there is a one-to-one correspondence between the irreducible representations of μ_k and the divisors $D_1,\dots,D_{k-1}$ [21, Corollary 10.3.11]. By [21, Equation (10.4.3)], the intersection matrix $(D_i\cdot D_j)_{1\leq i,j\leq k-1}$ is given by minus the Cartan matrix C of type A_{k-1}, i.e., one has

$$(D_i\cdot D_j)_{1\leq i,j\leq k-1} = -C = \begin{pmatrix} -2 & 1 & \cdots & 0\\ 1 & -2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -2\end{pmatrix} .$$

The surface X_k is an ALE space of type A_{k-1}.

Let U_i be the torus-invariant affine open subset of X_k which is a neighbourhood of the torus-fixed point p_i for $i=1,\dots,k$. Its coordinate ring is given by $\mathbb{C}[U_i] := \mathbb{C}[T_1^{2-i}\, T_2^{1-i}, T_1^{i-1}\, T_2^{i}]$ for $i=1,\dots,k$. By imposing the change of variables $T_1 = t_1^k$ and $T_2 = t_2\, t_1^{1-k}$, we have

$$\mathbb{C}[U_i] = \mathbb{C}[t_1^{k-i+1}\, t_2^{1-i}, t_1^{i-k}\, t_2^{i}] . \tag{5.1}$$

Define

$$\chi_1^i(t_1,t_2) = t_1^{k-i+1}\, t_2^{1-i} \qquad \text{and} \qquad \chi_2^i(t_1,t_2) = t_1^{i-k}\, t_2^{i} .$$

After identifying characters of T with one-dimensional T-modules, let $\varepsilon_j^{(i)}$ denote the equivariant first Chern class of χ_j^i for $i=1,\dots,k$ and $j=1,2$. Then

$$\varepsilon_1^{(i)}(\varepsilon_1,\varepsilon_2) = (k-i+1)\,\varepsilon_1 - (i-1)\,\varepsilon_2 \qquad \text{and} \qquad \varepsilon_2^{(i)}(\varepsilon_1,\varepsilon_2) = -(k-i)\,\varepsilon_1 + i\,\varepsilon_2 .$$

One can compactify the ALE space X_k to a normal projective toric surface $\bar{X}_k$ by adding a torus-invariant divisor $D_\infty \simeq \mathbb{P}^1$ such that for $k=2$ the surface $\bar{X}_2$ coincides with the second Hirzebruch surface $\mathbb{F}_2$. For $k\geq 3$ the surface $\bar{X}_k$ is singular, but one can associate with $\bar{X}_k$ its canonical toric stack $\mathscr{X}_k^{\mathrm{can}}$ which is a two-dimensional projective toric orbifold with Deligne-Mumford torus T and coarse moduli space $\pi_k^{\mathrm{can}}\colon \mathscr{X}_k^{\mathrm{can}} \to \bar{X}_k$. By *canonical* we mean that the locus over which π_k^{can} is not an isomorphism has non-positive dimension; for $k=2$ one has $\mathscr{X}_2^{\mathrm{can}} \simeq \mathbb{F}_2$. Consider the one-dimensional, torus-invariant, integral closed substack $\tilde{\mathscr{D}}_\infty := (\pi_k^{\mathrm{can}})^{-1}(D_\infty)_{\mathrm{red}} \subset \mathscr{X}_k^{\mathrm{can}}$. By performing the k-th root construction on $\mathscr{X}_k^{\mathrm{can}}$ along $\tilde{\mathscr{D}}_\infty$ to extend the automorphism group of a generic point of $\tilde{\mathscr{D}}_\infty$ by μ_k, we obtain a two-dimensional projective toric orbifold $\mathscr{X}_k$ with Deligne-Mumford torus T and coarse moduli space $\pi_k\colon \mathscr{X}_k \to \bar{X}_k$. The surface X_k is isomorphic to the open subset $\mathscr{X}_k \setminus \mathscr{D}_\infty$ of $\mathscr{X}_k$, where $\mathscr{D}_\infty := \pi_k^{-1}(D_\infty)_{\mathrm{red}}$. Let $\mathscr{D}_i := \pi_k^{-1}(D_i)_{\mathrm{red}}$ be the divisors in $\mathscr{X}_k$ corresponding to D_i for $i=1,\dots,k-1$. The classes

$$-\sum_{j=1}^{k-1} \left(C^{-1}\right)^{ij} \mathscr{D}_j$$

are integral for $i=1,\dots,k-1$, where the inverse of the Cartan matrix C is given by

$$\left(C^{-1}\right)^{ij} = \frac{i\,(k-j)}{k} \qquad \text{for} \quad i\leq j .$$

Denote by $\mathcal{R}_i$ the associated line bundles on $\mathscr{X}_k$; the restrictions of $\mathcal{R}_i$ to X_k are precisely the tautological line bundles of Kronheimer and Nakajima [32].

Proposition 5.2 ([14, Proposition 3.25]). *The Picard group* $\mathrm{Pic}(\mathscr{X}_k)$ *of* $\mathscr{X}_k$ *is freely generated over* $\mathbb{Z}$ *by* $\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_\infty)$ *and* $\mathcal{R}_i$ *with* $i = 1, \dots, k-1$.

The divisor $\mathscr{D}_\infty$ can be characterized as a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathbb{C}^* \times \mathscr{B}\mu_k$ and coarse moduli space $r_k \colon \mathscr{D}_\infty \to D_\infty$.

Proposition 5.3 ([14, Proposition 3.27]). *The divisor* $\mathscr{D}_\infty$ *is isomorphic as a toric Deligne-Mumford stack to the global toric quotient stack*

$$\left[\frac{\mathbb{C}^2 \setminus \{0\}}{\mathbb{C}^* \times \mu_k}\right] ,$$

where the group action is given in [14, Equation (3.28)].

Corollary 5.4 ([14, Corollary 3.29]). *The Picard group* $\mathrm{Pic}(\mathscr{D}_\infty)$ *is isomorphic to* $\mathbb{Z} \oplus \mathbb{Z}_k$. *It is generated by the line bundles* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *corresponding respectively to the characters*

$$\chi_1 : (t, \omega) \in \mathbb{C}^* \times \mu_k \longmapsto t \in \mathbb{C}^* \quad \textit{and} \quad \chi_2 : (t, \omega) \in \mathbb{C}^* \times \mu_k \longmapsto \omega \in \mathbb{C}^* .$$

For $j = 0, 1, \dots, k-1$ define the line bundles

$$\mathcal{O}_{\mathscr{D}_\infty}(j) = \begin{cases} \mathcal{L}_2^{\otimes j} & \text{for even } k , \\ \mathcal{L}_2^{\otimes j \, \frac{k+1}{2}} & \text{for odd } k . \end{cases}$$

Proposition 5.5 ([14, Corollary 3.34]). *The restrictions of the tautological line bundles* $\mathcal{R}_j$ *to* $\mathscr{D}_\infty$ *are given by*

$$\mathcal{R}_j\big|_{\mathscr{D}_\infty} \simeq \mathcal{O}_{\mathscr{D}_\infty}(j) .$$

Remark 5.6. In [14] the line bundles $\mathcal{O}_{\mathscr{D}_\infty}(j)$ are the line bundles $\mathcal{O}_{\mathscr{D}_\infty}(s, j)$ for $s = 0$. Indeed, one can prove that the degree of $\mathcal{O}_{\mathscr{D}_\infty}(j)$ is zero. Moreover, $\mathcal{O}_{\mathscr{D}_\infty}(0, j)$ can be endowed with a unitary flat connection associated with the j-th irreducible unitary representation ρ_j of $\mathbb{Z}_k$ for $j = 0, 1, \dots, k-1$ (cf. [23, Remark 6.5]). △

5.2 Rank one framed sheaves

Definition 5.7. Fix $j \in \{0, 1, \dots, k-1\}$. A *rank one* $(\mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(j))$*-framed sheaf* on $\mathscr{X}_k$ is a pair $(\mathcal{E}, \phi_\mathcal{E})$, where $\mathcal{E}$ is a torsion-free sheaf on $\mathscr{X}_k$ of rank one which is locally free in a neighbourhood of $\mathscr{D}_\infty$, and $\phi_\mathcal{E} \colon \mathcal{E}\big|_{\mathscr{D}_\infty} \xrightarrow{\sim} \mathcal{O}_{\mathscr{D}_\infty}(j)$ is an isomorphism. We call $\phi_\mathcal{E}$ a *framing* of $\mathcal{E}$. A *morphism* between $(\mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(j))$-framed sheaves $(\mathcal{E}, \phi_\mathcal{E})$ and $(\mathcal{G}, \phi_\mathcal{G})$ of rank one is a morphism $f \colon \mathcal{E} \to \mathcal{G}$ such that $\phi_\mathcal{G} \circ f\big|_{\mathscr{D}_\infty} = \phi_\mathcal{E}$. ⊘

Remark 5.8. By [14, Remark 4.3], the Picard group of $\mathscr{X}_k$ is isomorphic to the second singular cohomology group of $\mathscr{X}_k$ with integral coefficients via the first Chern class map c_1. Thus fixing the determinant line bundle of a coherent sheaf $\mathcal{E}$ on $\mathscr{X}_k$ is equivalent to fixing its first Chern class. △

Given a vector $\vec{u} = (u_1, \dots, u_{k-1}) \in \mathbb{Z}^{k-1}$, we denote by $\mathcal{R}^{\vec{u}}$ the line bundle $\bigotimes_{i=1}^{k-1} \mathcal{R}_i^{\otimes u_i}$ and by $\mathcal{R}_0$ the trivial line bundle $\mathcal{O}_{\mathscr{X}_k}$.

Lemma 5.9 ([14, Lemma 4.4]). *Let $(\mathcal{E},\phi_{\mathcal{E}})$ be a rank one $(\mathscr{D}_\infty,\mathcal{O}_{\mathscr{D}_\infty}(j))$-framed sheaf on $\mathscr{X}_k$. Then the determinant $\det(\mathcal{E})$ of $\mathcal{E}$ is of the form $\mathcal{R}^{\vec{u}}$, where the vector $\vec{u}\in\mathbb{Z}^{k-1}$ satisfies the condition*

$$\sum_{i=1}^{k-1} i\,u_i = j \mod k\ . \tag{5.10}$$

Remark 5.11. Set $\vec{v}:=C^{-1}\vec{u}$. Then Equation (5.10) implies the relations

$$k\,v_l = -l\,j \mod k$$

for $l=1,\dots,k-1$. Note that a component v_l is integral if and only if every component of $\vec{v}$ is integral. We subdivide the vectors $\vec{u}\in\mathbb{Z}^{k-1}$ according to Equation (5.10) as

$$\mathfrak{U}_j := \Big\{\vec{u}\in\mathbb{Z}^{k-1}\ \Big|\ \sum_{i=1}^{k-1} i\,u_i = j \mod k\Big\}\ .$$

Define now a bijective map by identifying a vector $\vec{u}\in\mathbb{Z}^{k-1}$ with $\sum_{i=1}^{k-1} u_i\,\mathrm{c}_1(\mathcal{R}_i)=\sum_{i=1}^{k-1} u_i\,\omega_i$ as

$$\psi:\ \vec{u}\in\mathbb{Z}^{k-1}\ \longmapsto\ \sum_{i=1}^{k-1} u_i\,\omega_i\in\mathfrak{P}\ .$$

It is natural to split this map according to the coset decomposition (3.17) as

$$\psi^{-1}(\mathfrak{Q}+\omega_j)=\mathfrak{U}_j\ ,$$

which means that $\psi(\vec{u}\,)$ for $\vec{u}\in\mathfrak{U}_j$ is naturally written as a sum of the fundamental weight ω_j and an element $\gamma_{\vec{u}}$ of the root lattice $\mathfrak{Q}$, which is given by

$$\gamma_{\vec{u}} := \sum_{i=1}^{k-1}\Big(\sum_{l=1}^{k-1}\big(C^{-1}\big)^{il}u_l-\big(C^{-1}\big)^{ij}\Big)\gamma_i=\sum_{i=1}^{k-1}\Big(v_i-\big(C^{-1}\big)^{ij}\Big)\gamma_i\ \in\ \mathfrak{Q}\ .$$

We write

$$\psi_j:=\psi\big|_{\mathfrak{U}_j}:\ \mathfrak{U}_j\ \longrightarrow\ \mathfrak{Q}+\omega_j\ . \tag{5.12}$$

$\triangle$

Following [14, Section 4], let $\mathcal{M}(\vec{u},n,j)$ be the fine moduli space parameterizing $(\mathscr{D}_\infty,\mathcal{O}_{\mathscr{D}_\infty}(j))$-framed sheaves of rank one on $\mathscr{X}_k$ with determinant line bundle $\mathcal{R}^{\vec{u}}$ and second Chern class $n\in\mathbb{Z}$; the vector $\vec{u}$ belongs to $\mathfrak{U}_j$. Let $p_{\mathscr{X}_k}:\mathscr{X}_k\times\mathcal{M}(\vec{u},n,j)\to\mathscr{X}_k$ be the projection. As explained in [14, Remark 4.7], by "fine" one means that there exists a *universal framed sheaf* $(\boldsymbol{\mathcal{E}},\boldsymbol{\phi}_{\boldsymbol{\mathcal{E}}})$, where $\boldsymbol{\mathcal{E}}$ is a coherent sheaf on $\mathcal{M}(\vec{u},n,j)\times\mathscr{X}_k$ which is flat over $\mathcal{M}(\vec{u},n,j)$, and $\boldsymbol{\phi}_{\boldsymbol{\mathcal{E}}}:\boldsymbol{\mathcal{E}}\to p_{\mathscr{X}_k}^*(\mathcal{O}_{\mathscr{D}_\infty}(j))$ is a morphism such that its restriction to $\mathcal{M}(\vec{u},n,j)\times\mathscr{D}_\infty$ is an isomorphism; the fibre over $[(\mathcal{E},\phi_{\mathcal{E}})]\in\mathcal{M}(\vec{u},n,j)$ is itself the $(\mathscr{D}_\infty,\mathcal{O}_{\mathscr{D}_\infty}(j))$-framed sheaf $(\mathcal{E},\phi_{\mathcal{E}})$ on $\mathscr{X}_k$. In the following we shall call $\boldsymbol{\mathcal{E}}$ the *universal sheaf.*

Theorem 5.13 ([14, Theorem 4.13]). *The moduli space $\mathcal{M}(\vec{u},n,j)$ is a smooth quasi-projective variety of dimension $2n$. The Zariski tangent space of $\mathcal{M}(\vec{u},n,j)$ at a point $[(\mathcal{E},\phi_{\mathcal{E}})]$ is $\mathrm{Ext}^1(\mathcal{E},\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))$.*

As explained in [14, Section 4.3], the Hilbert scheme of n points $\mathrm{Hilb}^n(X_k)$ of X_k is isomorphic to $\mathcal{M}(\vec{u}, n, j)$ for any $\vec{u} \in \mathfrak{U}_j$. For this, let $\imath : X_k \hookrightarrow \mathscr{X}_k$ be the inclusion morphism. If Z is a point of $\mathrm{Hilb}^n(X_k)$ and $\vec{y} \in \mathbb{Z}^{k-1}$, then the coherent sheaf $\mathcal{E} := \imath_*(\mathcal{I}_Z) \otimes \mathcal{R}^{e_j - C\vec{y}}$ is a rank one torsion-free sheaf on $\mathscr{X}_k$ with a framing $\phi_\mathcal{E}$ induced by the canonical isomorphism $\mathcal{R}^{e_j - C\vec{y}}|_{\mathscr{D}_\infty} \xrightarrow{\sim} \mathcal{O}_{\mathscr{D}_\infty}(j)$ such that $(\mathcal{E}, \phi_\mathcal{E})$ is a rank one $(\mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(j))$-framed sheaf with determinant line bundle $\mathcal{R}^{\vec{u}}$, where $\vec{u} := \boldsymbol{e}_j - C\vec{y}$, and second Chern class n. Thus Z induces a point $[(\mathcal{E}, \phi_\mathcal{E})]$ in $\mathcal{M}(\vec{u}, n, j)$. This defines an inclusion morphism

$$\tilde{\imath}_{\vec{u},n,j} : \mathrm{Hilb}^n(X_k) \hookrightarrow \mathcal{M}(\vec{u}, n, j)$$

which is an isomorphism of fine moduli spaces by [14, Proposition 4.16].

Remark 5.14. In [33] it is shown that the Hilbert scheme of points $\mathrm{Hilb}^n(X_k)$ is isomorphic to a Nakajima quiver variety of type $\widehat{A}_{k-1}$ with suitable dimension vectors. Thus $\mathcal{M}(\vec{u}, n, j)$ is a quiver variety. △

5.3 Equivariant cohomology

We define a T-action on $\mathcal{M}(\vec{u}, n, j)$ in the following way (cf. [14, Section 4.6]). For $(t_1, t_2) \in T$ let $F_{(t_1,t_2)}$ be the automorphism of $\mathscr{X}_k$ induced by the torus action; then the T-action is given by

$$(t_1, t_2) \triangleright \big[(\mathcal{E}, \phi_\mathcal{E})\big] := \big[\big((F^{-1}_{(t_1,t_2)})^*(\mathcal{E}) \,,\, \phi'_\mathcal{E}\big)\big] \ ,$$

where $\phi'_\mathcal{E}$ is the composition of isomorphisms

$$\phi'_\mathcal{E} : \ \big(F^{-1}_{(t_1,t_2)}\big)^*\mathcal{E}\big|_{\mathscr{D}_\infty} \xrightarrow{(F^{-1}_{(t_1,t_2)})^*(\phi_\mathcal{E})} \big(F^{-1}_{(t_1,t_2)}\big)^*\mathcal{O}_{\mathscr{D}_\infty}(j) \longrightarrow \mathcal{O}_{\mathscr{D}_\infty}(j) \ ;$$

here the last arrow is given by the T-equivariant structure induced on $\mathcal{O}_{\mathscr{D}_\infty}(j)$ by restriction of the torus action of $\mathscr{X}_k$ to $\mathscr{D}_\infty$. Note that the T-action on X_k naturally lifts to $\mathrm{Hilb}^n(X_k)$ and the isomorphism $\tilde{\imath}_{\vec{u},n,j}$ is equivariant with respect to these torus actions.

Proposition 5.15 ([14, Proposition 4.22])**.** *For a T-fixed point $[(\mathcal{E}, \phi_\mathcal{E})] \in \mathcal{M}(\vec{u}, n, j)^T$ the underlying sheaf is of the form $\mathcal{E} = \imath_*(\mathcal{I}_Z) \otimes \mathcal{R}^{\vec{u}}$, where $\mathcal{I}_Z$ is the ideal sheaf of a T-fixed zero-dimensional subscheme Z of X_k.*

Remark 5.16. Let $[(\mathcal{E}, \phi_\mathcal{E})]$ be a T-fixed point of $\mathcal{M}(\vec{u}, n, j)$, with $\mathcal{E} = \imath_*(\mathcal{I}_Z) \otimes \mathcal{R}^{\vec{u}}$. The T-fixed subscheme Z of X_k of length n is a disjoint union of T-fixed subschemes Z_i for $i = 1, \ldots, k$ supported at the T-fixed points p_i with $\sum_{i=1}^k \mathrm{length}_{p_i}(Z_i) = n$. Put $n_i = \mathrm{length}_{p_i}(Z_i)$. Since p_i is the T-fixed point of the smooth affine toric surface $U_i \simeq \mathbb{C}^2$, as explained in Section 4.1 the T-fixed subscheme $Z_i \in \mathrm{Hilb}^{n_i}(U_i)$ corresponds to a Young tableau Y^i of weight $|Y^i| = n_i$ for $i = 1, \ldots, k$. Thus the T-fixed point Z corresponds to a k-tuple of Young tableaux $\vec{Y} = (Y^1, \ldots, Y^k)$ with $|\vec{Y}| := \sum_{i=1}^k |Y^i| = n$. Hence we can parametrize the point $[(\mathcal{E}, \phi_\mathcal{E})]$ by the pair $(\vec{Y}, \vec{u}\,)$ which we call the *combinatorial datum* of $[(\mathcal{E}, \phi_\mathcal{E})]$. △

Consider the T-equivariant cohomology of the moduli spaces $\mathcal{M}(\vec{u}, n, j)$ and set

$$\mathbb{W}_{\vec{u},n,j} := H^*_T\big(\mathcal{M}(\vec{u}, n, j)\big)_{\mathrm{loc}} \ .$$

We endow $\mathbb{W}_{\vec{u},n,j}$ with the nondegenerate $\mathbb{C}(\varepsilon_1, \varepsilon_2)$-valued bilinear form

$$\langle A, B\rangle_{\mathbb{W}_{\vec{u},n,j}} := (-1)^n \, p^!_{\vec{u},n,j} \, \big(\imath^!_{\vec{u},n,j}\big)^{-1}(A \cup B) \ ,$$

where $p_{\vec{u},n,j}$ is the projection from $\mathcal{M}(\vec{u},n,j)$ to a point and $\imath_{\vec{u},n,j}\colon \mathcal{M}(\vec{u},n,j)^T \hookrightarrow \mathcal{M}(\vec{u},n,j)$ is the inclusion of the fixed-point locus. Thus for $\vec{u} \in \mathfrak{U}_j$ we define

$$\mathbb{W}_{\vec{u},j} := \bigoplus_{n\geq 0} \mathbb{W}_{\vec{u},n,j} \ , \tag{5.17}$$

and the total equivariant cohomology

$$\mathbb{W}_j := \bigoplus_{\vec{u}\in\mathfrak{U}_j} \bigoplus_{n\geq 0} \mathbb{W}_{\vec{u},n,j}$$

which is an infinite-dimensional vector space over the field $\mathbb{C}(\varepsilon_1,\varepsilon_2)$ endowed with the nondegenerate $\mathbb{C}(\varepsilon_1,\varepsilon_2)$-valued bilinear form $\langle -,-\rangle_{\mathbb{W}_j}$ induced by the symmetric bilinear forms $\langle -,-\rangle_{\mathbb{W}_{\vec{u},n,j}}$.

Let us denote by $[\vec{Y},\vec{u}\,]$ the equivariant cohomology class, defined similarly to (4.2), associated with the T-fixed point $[(\mathcal{E},\phi_{\mathcal{E}})]$ with combinatorial datum $(\vec{Y},\vec{u}\,)$. By the localization theorem, the classes $[\vec{Y},\vec{u}\,]$ with $\vec{u}\in\mathfrak{U}_j$ form a $\mathbb{C}(\varepsilon_1,\varepsilon_2)$-basis of $\mathbb{W}_j$.

6 Representations of $\widehat{\mathfrak{gl}}_k$

6.1 Overview

The results collected so far imply the following result.

Proposition 6.1. *There is an isomorphism*

$$\begin{aligned}\Psi_j\,:\,\mathbb{W}_j \ \xrightarrow{\ \sim\ }\ & \bigoplus_{\vec{u}\in\mathfrak{U}_j}\bigoplus_{n\geq 0} H_T^*\big(\mathrm{Hilb}^n(X_k)\big)_{\mathrm{loc}} \\ &\simeq \Big(\bigoplus_{n\geq 0} H_T^*\big(\mathrm{Hilb}^n(X_k)\big)_{\mathrm{loc}}\Big)\otimes\mathbb{C}(\varepsilon_1,\varepsilon_2)[\mathfrak{U}_j] \\ &\xrightarrow{\ \sim\ }\ \Big(\bigoplus_{n\geq 0} H_T^*\big(\mathrm{Hilb}^n(X_k)\big)_{\mathrm{loc}}\Big)\otimes\mathbb{C}(\varepsilon_1,\varepsilon_2)[\mathfrak{Q}+\omega_j]\ ,\end{aligned} \tag{6.2}$$

where the first arrow is induced by the morphisms $\tilde{\imath}^*_{\vec{u},n,j}$ while the last arrow is induced by the map ψ_j introduced in (5.12)*. There is also an isomorphism*

$$\Psi := \bigoplus_{j=0}^{k-1}\Psi_j\,:\,\mathbb{W}\ \xrightarrow{\ \sim\ }\ \bigoplus_{j=0}^{k-1}\bigoplus_{n\geq 0} H_T^*\big(\mathrm{Hilb}^n(X_k)\big)_{\mathrm{loc}}\otimes\mathbb{C}(\varepsilon_1,\varepsilon_2)[\mathfrak{Q}+\omega_j]\ ,$$

where $\mathbb{W} := \bigoplus_{j=0}^{k-1}\mathbb{W}_j$.

In this section we first study the equivariant cohomology of $\mathrm{Hilb}^n(X_k)$ and construct over it an action of the sum (identifying central elements) $\mathfrak{h}_{\mathbb{C}(\varepsilon_1,\varepsilon_2)}\oplus\mathfrak{h}_{\mathbb{C}(\varepsilon_1,\varepsilon_2),\mathfrak{Q}}$. Then we use the Frenkel-Kac construction (Theorem 3.27) to obtain an action of $\widehat{\mathfrak{gl}}_k=\mathfrak{h}_{\mathbb{C}(\varepsilon_1,\varepsilon_2)}\oplus\widehat{\mathfrak{sl}}_k$ on $\mathbb{W}_j$ for $j=0,1,\dots,k-1$.

6.2 Equivariant cohomology of $\mathrm{Hilb}^n(X_k)$

In this subsection we derive some results concerning the equivariant cohomology of the Hilbert schemes $\mathrm{Hilb}^n(X_k)$ by generalizing similar results of [55, Section 2] (see also [39, Section 2]).

As discussed in Remark 5.16, a T-fixed point $Z \in \mathrm{Hilb}^n(X_k)$ corresponds to a k-tuple $(Z_1, \dots, Z_k)$ where Z_i is a T-fixed point of $\mathrm{Hilb}^{n_i}(U_i)$ for $i = 1, \dots, k$ with $\sum_{i=1}^k n_i = n$, or equivalently to a k-tuple $\vec{Y} = (Y^1, \dots, Y^k)$ of Young tableaux with $|\vec{Y}| := \sum_{i=1}^k |Y^i| = n$. The following result is straightforward to prove.

Lemma 6.3. *Let Z be a T-fixed point of $\mathrm{Hilb}^n(X_k)$. Then there is a T-equivariant isomorphism*

$$T_Z \mathrm{Hilb}^n(X_k) \simeq \bigoplus_{i=1}^k T_{Z_i} \mathrm{Hilb}^{n_i}(U_i) ,$$

where $Z = \bigsqcup_{i=1}^k Z_i$ and n_i is the length of Z_i at p_i for $i = 1, \dots, k$.

By Lemma 6.3 we get

$$\mathrm{ch}_T\big(T_Z \mathrm{Hilb}^n(X_k)\big) = \sum_{i=1}^k \mathrm{ch}_T\big(T_{Z_i} \mathrm{Hilb}^{n_i}(U_i)\big) .$$

By using the description (5.1) of the coordinate ring $\mathbb{C}[U_i]$ of U_i, one computes the equivariant Chern characters

$$\mathrm{ch}_T\big(T_{Z_i} \mathrm{Hilb}^{n_i}(U_i)\big) = \sum_{s \in Y^i} \big(\mathrm{e}^{\,(L(s)+1)\,\varepsilon_1^{(i)} - A(s)\,\varepsilon_2^{(i)}} + \mathrm{e}^{\,-L(s)\,\varepsilon_1^{(i)} + (A(s)+1)\,\varepsilon_2^{(i)}} \big) .$$

From now on we identify a torus-fixed point Z of $\mathrm{Hilb}^n(X_k)$ with its k-tuple $\vec{Y}$ of Young tableaux.

Let $\vec{Y} = (Y^1, \dots, Y^k)$ be a torus-fixed point. Define

$$\mathrm{eu}_+(\vec{Y}) := \prod_{i=1}^k \prod_{s \in Y^i} \Big(\big(L(s)+1\big)\,\varepsilon_1^{(i)} - A(s)\,\varepsilon_2^{(i)} \Big) ,$$

$$\mathrm{eu}_-(\vec{Y}) := \prod_{i=1}^k \prod_{s \in Y^i} \Big(L(s)\,\varepsilon_1^{(i)} - \big(A(s)+1\big)\,\varepsilon_2^{(i)} \Big) .$$

Then the equivariant Euler class of the tangent bundle at the fixed point $\vec{Y}$ is given by

$$\mathrm{eu}_T\left(T_{\vec{Y}} \mathrm{Hilb}^n(X_k)\right) = (-1)^n \ \mathrm{eu}_+(\vec{Y}) \ \mathrm{eu}_-(\vec{Y}) .$$

6.2.1 Equivariant basis I: Torus-fixed points

Let $\vec{Y}$ be a k-tuple of Young tableaux corresponding to a fixed point in $\mathrm{Hilb}^n(X_k)$ and $[\vec{Y}]$ the equivariant cohomology class defined similarly to (4.2). By the projection formula we get

$$[\vec{Y}] \cup [\vec{Y}'] = \delta_{\vec{Y},\vec{Y}'} \ \mathrm{eu}_T\left(T_{\vec{Y}} \mathrm{Hilb}^n(X_k)\right)[Y] = (-1)^n \ \delta_{\vec{Y},\vec{Y}'} \ \mathrm{eu}_+(\vec{Y}) \ \mathrm{eu}_-(\vec{Y})[\vec{Y}] .$$

Denote

$$\imath_n := \bigoplus_{\vec{Y}\in \mathrm{Hilb}^n(X_k)^T} \imath_{\vec{Y}} \,:\, \mathrm{Hilb}^n(X_k)^T \longrightarrow \mathrm{Hilb}^n(X_k)\ .$$

In analogy to Equation (4.3), define the bilinear form

$$\begin{aligned} \langle -,-\rangle_{\mathbb{H}_n} \,:\, \mathbb{H}_n \times \mathbb{H}_n &\longrightarrow \mathbb{C}(\varepsilon_1,\varepsilon_2)\ , \\ (A,B) &\longmapsto (-1)^n\, p_n^!\, \big(\imath_n^!\big)^{-1}(A\cup B) \end{aligned} \tag{6.4}$$

where $\mathbb{H}_n := H_T^*(\mathrm{Hilb}^n(X_k))_{\mathrm{loc}}$.

As in Section 4.1, for any class $[\vec{Y}] \in H_T^{4n}(\mathrm{Hilb}^n(X_k))$ we define a distinguished class

$$[\alpha_{\vec{Y}}] := \frac{[\vec{Y}]}{\mathrm{eu}_+(\vec{Y})} \ \in\ H_T^{2n}\big(\mathrm{Hilb}^n(X_k)\big)_{\mathrm{loc}}\ .$$

Then by the same computation as in Equation (4.5) we get

$$\big\langle [\alpha_{\vec{Y}}]\,,\,[\alpha_{\vec{Y}'}]\big\rangle_{\mathbb{H}_n} = \delta_{\vec{Y},\vec{Y}'}\,\frac{\mathrm{eu}_-(\vec{Y})}{\mathrm{eu}_+(\vec{Y})} = \delta_{\vec{Y},\vec{Y}'}\,\prod_{i=1}^k\,\prod_{s\in Y^i} \frac{L(s)\,\beta_i + A(s)+1}{\big(L(s)+1\big)\,\beta_i + A(s)}\ , \tag{6.5}$$

where analogously to (4.6) we defined

$$\beta_i := -\frac{\varepsilon_1^{(i)}}{\varepsilon_2^{(i)}}\ .$$

Note that when $n=1$, $\vec{Y}$ is just a fixed point $p_i \in X_k^T$ with $i=1,\dots,k$. Thus we have

$$\mathrm{eu}_+(p_i) = \varepsilon_1^{(i)} = (k-i+1)\,\varepsilon_1 - (i-1)\,\varepsilon_2 \qquad \text{and} \qquad \mathrm{eu}_-(p_i) = -\varepsilon_2^{(i)} = (k-i)\,\varepsilon_1 - i\,\varepsilon_2\ ,$$

and therefore

$$\beta_i = \frac{\mathrm{eu}_+(p_i)}{\mathrm{eu}_-(p_i)}\ .$$

If for $i=1,\dots,k$ we define $[\alpha_i] := [\alpha_{p_i}]$, then we get

$$\big\langle [\alpha_i]\,,\,[\alpha_j]\big\rangle_{\mathbb{H}_1} = \beta_i^{-1}\,\delta_{ij} \ \in\ \mathbb{C}(\varepsilon_1,\varepsilon_2)\ .$$

By the localization theorem and Equation (6.5), the classes $[\alpha_{\vec{Y}}]$ with $|\vec{Y}|=n$ form a $\mathbb{C}(\varepsilon_1,\varepsilon_2)$-linear basis of $\mathbb{H}_n$. Hence the bilinear form (6.4) is nondegenerate; it extends to give a nondegenerate symmetric bilinear form $\langle -,-\rangle_{\mathbb{H}}$ on the total equivariant cohomology $\mathbb{H} := \bigoplus_{n\ge 0}\mathbb{H}_n$ of the Hilbert schemes of points on X_k.

Remark 6.6. Let $i\in\{1,\dots,k\}$. By the localization theorem, the $\mathbb{C}(\varepsilon_1,\varepsilon_2)$-linear subspace of $\mathbb{H}$ generated by all classes $[\vec{Y}]$ associated to fixed points $\vec{Y}=(Y^1,\dots,Y^k)$ such that $Y^j=\emptyset$ for every $j\in\{1,\dots,k\}$ with $j\neq i$ is isomorphic to

$$\bigoplus_{m\ge 0} H_T^*\big(\mathrm{Hilb}^m(U_i)\big) \otimes_{\mathbb{C}[\varepsilon_1^{(i)},\varepsilon_2^{(i)}]} \mathbb{C}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)}\big)\ . \tag{6.7}$$

△

Note that $\mathbb{C}\big[\varepsilon_1^{(i)},\varepsilon_2^{(i)}\big]=\mathbb{C}[\varepsilon_1,\varepsilon_2]$ and $\mathbb{C}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)}\big)=\mathbb{C}(\varepsilon_1,\varepsilon_2)$. Analogously to what we did for $\mathbb{C}^2$, we can thus define

$$\mathbb{H}_{U_i,m} := H_T^*\big(\mathrm{Hilb}^m(U_i)\big)_{\mathrm{loc}} \qquad \text{and} \qquad \mathbb{H}_{U_i} := \bigoplus_{m\geq 0} \mathbb{H}_{U_i,m} \ .$$

By the localization theorem, there exists a $\mathbb{C}(\varepsilon_1,\varepsilon_2)$-linear isomorphism

$$\Omega\,:\ \mathbb{H} \ \xrightarrow{\ \sim\ }\ \bigotimes_{i=1}^k \mathbb{H}_{U_i} \ . \tag{6.8}$$

In particular, for a fixed point $\vec{Y}=(Y^1,\dots,Y^k)$ we have

$$\Omega\,:\ [\alpha_{\vec{Y}}] \ \longmapsto\ [\alpha_{Y^1}]\otimes\cdots\otimes[\alpha_{Y^k}] \ .$$

The isomorphism Ω interwines the bilinear forms $\langle -,-\rangle_{\mathbb{H}}$ and $\prod_{i=1}^k \langle -,-\rangle_i$, where $\langle -,-\rangle_i$ is the symmetric bilinear form on $\mathbb{H}_{U_i}$ defined analogously to (4.3). In a similar way, there is a $\mathbb{C}(\varepsilon_1,\varepsilon_2)$-linear isomorphism

$$\Omega_k\,:\ \mathbb{H}_1 \ \xrightarrow{\ \sim\ }\ \bigoplus_{i=1}^k \mathbb{H}_{U_i,1} \ . \tag{6.9}$$

In this case $\Omega_k\colon[\alpha_i]\ \mapsto\ (0,\dots,[\alpha_i],\dots,0)$, where the class $[\alpha_i]$ on the left-hand side belongs to $\mathbb{H}_1=H_T^*(X_k)_{\mathrm{loc}}$ while on the right-hand side it belongs to $\mathbb{H}_{U_i,1}$ as defined in Section 4.1. The isomorphism Ω_k also intertwines the symmetric bilinear forms.

6.2.2 Equivariant basis II: Torus-invariant divisors

Let $[D_i]_T$ be the class in $\mathbb{H}_1=H_T^*(X_k)_{\mathrm{loc}}$ given by the T-invariant divisor D_i for $i=0,1,\dots,k$. For $i=1,\dots,k-1$, we have

$$[D_i]_T = \frac{[p_i]}{\mathrm{eu}_T(T_{p_i}D_i)}+\frac{[p_{i+1}]}{\mathrm{eu}_T(T_{p_{i+1}}D_i)} = \frac{[p_i]}{\varepsilon_2^{(i)}}+\frac{[p_{i+1}]}{\varepsilon_1^{(i+1)}} = -\beta_i\,[\alpha_i]+[\alpha_{i+1}] \ . \tag{6.10}$$

Thus for $i,j=1,\dots,k-1$ we obtain the pairings

$$\big\langle [D_i]_T\,,\,[D_j]_T\big\rangle_{\mathbb{H}_1} = \begin{cases} 2\,, & i=j\,,\\ -1\,, & |i-j|=1\,,\\ 0\,, & \text{otherwise}\,. \end{cases} \tag{6.11}$$

By applying the localization theorem to $[D_0]_T$ and $[D_k]_T$ we further obtain

$$[D_0]_T=\frac{[p_1]}{k\,\varepsilon_1}=\frac{[p_1]}{\varepsilon_1^{(1)}}=[\alpha_1] \qquad \text{and} \qquad [D_k]_T=\frac{[p_k]}{k\,\varepsilon_2}=\frac{[p_k]}{\varepsilon_2^{(k)}}=-\beta_k\,[\alpha_k] \ .$$

By using these expressions, one can straightforwardly obtain the pairings

$$\big\langle [D_0]_T\,,\,[D_i]_T\big\rangle_{\mathbb{H}_1} = \begin{cases} \beta_1^{-1}\,, & i=0\,,\\ -1\,, & i=1\,,\\ 0\,, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \big\langle [D_k]_T\,,\,[D_i]_T\big\rangle_{\mathbb{H}_1} = \begin{cases} \beta_k\,, & i=k\,,\\ -1\,, & i=k-1\,,\\ 0\,, & \text{otherwise}\,. \end{cases}$$

Now we can relate the classes $[\alpha_i]$ for $i = 1, \ldots, k$ to the classes $[D_j]_T$ for $j = 0, 1, \ldots, k$. By using Equation (6.10), for $i = 2, \ldots, k$ one obtains

$$[\alpha_i] = \sum_{j=0}^{i-2} \Big(\prod_{s=j+1}^{i-1} \beta_s \Big) [D_j]_T + [D_{i-1}]_T \ . \tag{6.12}$$

Since $\mathrm{eu}_+(p_l) = \mathrm{eu}_-(p_{l-1})$ for $l = 2, \ldots, k$, we get

$$\prod_{s=j+1}^{i-1} \beta_s = \frac{\mathrm{eu}_+(p_{j+1})}{\mathrm{eu}_-(p_{i-1})} \ .$$

By using the definition of $[\alpha_k]$ and Equation (6.12) for $i = k$ we obtain

$$-\beta_k^{-1} [D_k]_T = [\alpha_k] = \sum_{j=0}^{k-1} \frac{\mathrm{eu}_+(p_{j+1})}{\mathrm{eu}_-(p_{k-1})} [D_j]_T \ .$$

If we formally put $\mathrm{eu}_+(p_{k+1}) := \mathrm{eu}_-(p_k)$, we can reformulate this equation as

$$\sum_{j=0}^{k} \mathrm{eu}_+(p_{j+1}) \ [D_j]_T = 0 \ , \tag{6.13}$$

and in particular for all $i = 0, 1, \ldots, k$ we have

$$[D_i]_T = - \sum_{\substack{j=0 \\ j \neq i}}^{k} \frac{\mathrm{eu}_+(p_{j+1})}{\mathrm{eu}_+(p_{i+1})} [D_j]_T \ .$$

Remark 6.14. If the action is antidiagonal, i.e., $t = t_1 = t_2^{-1}$, Equation (6.13) implies that $\sum_{j=0}^{k} [D_j]_T = 0$. △

As the classes $[\alpha_1], \ldots, [\alpha_k]$ form a $\mathbb{C}(\varepsilon_1, \varepsilon_2)$-linear basis of $\mathbb{H}_1$, by Equations (6.12) and (6.13) the classes

$$\big\{ [D_0]_T, [D_1]_T, \ldots, [D_{k-1}]_T \big\} \qquad \text{and} \qquad \big\{ [D_1]_T, [D_2]_T, \ldots, [D_k]_T \big\} \tag{6.15}$$

are also $\mathbb{C}(\varepsilon_1, \varepsilon_2)$-linear bases for $\mathbb{H}_1$. Under the isomorphism Ω_k of Equation (6.9), we have

$$\Omega_k : \ [D_i]_T \longmapsto -\beta_i \big(0, \ldots, 0, [\alpha_i], 0, \ldots, 0\big) + \big(0, \ldots, 0, [\alpha_{i+1}], 0, \ldots, 0\big)$$

for $i = 1, \ldots, k-1$, together with a similar description for $[D_0]_T$ and $[D_k]_T$.

6.3 Heisenberg algebras

Let m be a positive integer and Y a torus-invariant closed curve in X_k. Define the correspondences

$$Y_{n,m} := \big\{ (Z, Z') \in \mathrm{Hilb}^{n+m}(X_k) \times \mathrm{Hilb}^n(X_k) \ \big| \ Z' \subset Z \ , \ \mathrm{supp}(\mathcal{I}_{Z'}/\mathcal{I}_Z) = \{y\} \subset Y \big\} \ .$$

Let q_1 and q_2 be the projections of $\mathrm{Hilb}^{n+m}(X_k) \times \mathrm{Hilb}^n(X_k)$ to the two factors respectively. We define the linear operator $\mathfrak{p}_{-m}([Y]_T) \colon \mathbb{H} \to \mathbb{H}$ which acts on $A \in \mathbb{H}_n$ as $\mathfrak{p}_{-m}([Y]_T)(A) := q_1^! \big(q_2^*(A) \cup [Y_{n,m}]_T\big) \in \mathbb{H}_{n+m}$. This definition is well-posed because the restriction of q_1 to $Y_{n,m}$ is proper. Since the bilinear form $\langle -, - \rangle_{\mathbb{H}}$ is nondegenerate on $\mathbb{H}$, we can define $\mathfrak{p}_m([Y]_T)$ to be the adjoint operator of $\mathfrak{p}_{-m}([Y]_T)$. By using one of the two bases in (6.15), we extend by linearity in α to obtain the linear operator $\mathfrak{p}_m(\alpha)$ for every $\alpha \in \mathbb{H}_1 = H_T^*(X_k)_{\mathrm{loc}}$.

Theorem 6.16 (see [55, 39])**.** *The linear operators $\mathfrak{p}_m(\alpha)$, where $m \in \mathbb{Z} \setminus \{0\}$ and $\alpha \in H^*_T(X_k)_{\mathrm{loc}}$, satisfy the Heisenberg commutation relations*

$$\big[\mathfrak{p}_m(\alpha), \mathfrak{p}_n(\beta)\big] = m\, \delta_{m,-n}\, \langle \alpha, \beta\rangle_{\mathbb{H}_1}\, \mathrm{id} \qquad \textit{and} \qquad \big[\mathfrak{p}_m(\alpha), \mathrm{id}\big] = 0\ .$$

The vector space $\mathbb{H}$ is the Fock space of the Heisenberg algebra $\mathfrak{h}_{\mathbb{H}_1}$ modelled on $\mathbb{H}_1 = H^*_T(X_k)_{\mathrm{loc}}$ with highest weight vector the unit element $|0\rangle$ in $H^0_T(\mathrm{Hilb}^0(X_k))_{\mathrm{loc}}$.

6.3.1 Heisenberg algebra of rank k

Let $i \in \{1, \dots, k\}$. Consider the Heisenberg algebra $\mathfrak{h}_i$ over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ generated by the operators

$$\mathfrak{p}^i_{-m} := \mathfrak{p}_{-m}([\alpha_i]) \qquad \text{and} \qquad \mathfrak{p}^i_m := \mathfrak{p}_m([\alpha_i])$$

for $m \in \mathbb{Z}_{>0}$. By Theorem 6.16, the commutation relations are

$$\big[\mathfrak{p}^i_m, \mathfrak{p}^j_n\big] = m\, \delta_{m,-n}\, \delta_{ij} \left\langle [\alpha_i]\, ,\, [\alpha_i]\right\rangle_{\mathbb{H}} \mathrm{id} = m\, \delta_{m,-n}\, \delta_{ij}\, \beta_i^{-1}\, \mathrm{id}\ .$$

Since $\{[\alpha_1], \dots, [\alpha_k]\}$ is a $\mathbb{C}(\varepsilon_1, \varepsilon_2)$-linear basis of $\mathbb{H}_1$, the Heisenberg algebra $\mathfrak{h}_{\mathbb{H}_1}$ is generated by $\mathfrak{p}^i_m$ for $i = 1, \dots, k$ and $m \in \mathbb{Z} \setminus \{0\}$.

Let $\mathbb{H}_{U_i}$ be the $\mathbb{C}(\varepsilon_1, \varepsilon_2)$-linear subspace of $\mathbb{H}$ introduced in Section 6.2.1. Then by Theorem 4.8 $\mathbb{H}_{U_i}$ is the Fock space for the Heisenberg algebra $\mathfrak{h}_i$ for any $i \in \{1, \dots, k\}$; therefore the $\mathbb{C}(\varepsilon_1, \varepsilon_2)$-vector space $\mathbb{H}_{U_i}$ is generated by the elements $\mathfrak{p}^i_\lambda|0\rangle$ where $\mathfrak{p}^i_\lambda := \prod_{l \geq 1} (\mathfrak{p}^i_{-l})^{m_l}$ for a partition $\lambda = (1^{m_1}\, 2^{m_2}\, \cdots)$. One can show that

$$\left\langle \mathfrak{p}^i_\lambda|0\rangle\, ,\, \mathfrak{p}^i_\mu|0\rangle\right\rangle_{\mathbb{H}_{U_i}} = \delta_{\lambda,\mu}\, z_\lambda\, \beta_i^{-\ell(\lambda)}\ .$$

On the algebra $\Lambda_{\mathbb{C}(\varepsilon_1,\varepsilon_2)}$ of symmetric functions over the field $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ we introduce the Jack inner product (2.1) with parameter β_i. We shall denote with Λ_{β_i} the algebra $\Lambda_{\mathbb{C}(\varepsilon_1,\varepsilon_2)}$ endowed with the symmetric bilinear form $\langle -, -\rangle_{\beta_i}$. Thus by the isomorphism (6.7) and Theorem 4.11 there exists an isomorphism of $\mathbb{C}(\varepsilon_1, \varepsilon_2)$-vector spaces

$$\Phi_i :\ \mathbb{H}_{U_i} \ \xrightarrow{\ \sim\ }\ \Lambda_{\beta_i}\ , \qquad \mathfrak{p}^i_\lambda|0\rangle\ \longmapsto\ p_\lambda\ ,$$

which intertwines the symmetric bilinear forms $\langle -, -\rangle_i$ and $\langle -, -\rangle_{\beta_i}$. For $m > 0$ the operator $\mathfrak{p}^i_{-m}$ acts as multiplication by p_m on Λ_{β_i} while its adjoint $\mathfrak{p}^i_m$ with respect to the symmetric bilinear form $\langle -, -\rangle_i$ acts as $m\, \beta_i^{-1}\, \frac{\partial}{\partial p_m}$.

By Theorem 4.11 we can also determine how Φ_i acts on the $\mathbb{C}(\varepsilon_1, \varepsilon_2)$-linear basis $\{[\alpha_{\vec{Y}}]\}$ of $\mathbb{H}_{U_i}$, where $\vec{Y} = (Y^1, \dots, Y^k)$ is a fixed point such that $Y^j = \emptyset$ for every $j \in \{1, \dots, k\}$ with $j \neq i$.

Proposition 6.17. *Let $\vec{Y} = (Y^1, \dots, Y^k)$ be a fixed point such that $Y^j = \emptyset$ for every $j \in \{1, \dots, k\}$ with $j \neq i$. Then*

$$\Phi_i([\alpha_{\vec{Y}}]) = J_{\lambda_i}(x; \beta_i^{-1})\ ,$$

where $Y_{\lambda_i} := Y^i$.

Define $\Lambda_{\vec{\beta}} = \bigotimes_{i=1}^k \Lambda_{\beta_i}$ endowed with the symmetric bilinear form $\langle p, q\rangle_{\vec{\beta}} := \prod_{i=1}^k \langle p_i, q_i\rangle_{\beta_i}$ for $p = p_1 \otimes \cdots \otimes p_k$ and $q = q_1 \otimes \cdots \otimes q_k$ in $\Lambda_{\vec{\beta}}$. For a k-tuple of Young tableaux $\vec{Y}$, define in $\mathcal{U}(\mathfrak{h}_{\mathbb{H}_1})$ the operators $\mathfrak{p}_{\vec{Y}} = \prod_{i=1}^k \mathfrak{p}^i_{\lambda_i}$. We have thus proven the following result.

Theorem 6.18. *There exists a $\mathbb{C}(\varepsilon_1,\varepsilon_2)$-linear isomorphism*

$$\Phi := \bigotimes_{i=1}^{k} \Phi_i : \mathbb{H} \longrightarrow \Lambda_{\vec{\beta}}$$

preserving bilinear forms such that

$$\Phi\left(\mathfrak{p}_{\vec{Y}}|0\rangle\right) = p_{\lambda_1} \otimes \cdots \otimes p_{\lambda_k} \qquad \text{and} \qquad \Phi\left([\alpha_{\vec{Y}}]\right) = J_{\lambda_1}(x;\beta_1^{-1}) \otimes \cdots \otimes J_{\lambda_k}(x;\beta_k^{-1}) \ .$$

Via the isomorphism Φ, the operators $\mathfrak{p}^i_m$ act on $\Lambda_{\vec{\beta}}$ as multiplication by p_{-m} on the i-th factor for $m<0$ and as the derivation $m\,\beta_i^{-1}\,\frac{\partial}{\partial p_m}$ on the i-th factor for $m>0$.

6.3.2 Lattice Heisenberg algebra of type A_{k-1}

Let us now define

$$\mathfrak{q}^i_{-m} := \mathfrak{p}_{-m}([D_i]_T) \qquad \text{and} \qquad \mathfrak{q}^i_m := \mathfrak{p}_m([D_i]_T)$$

for $m\in\mathbb{Z}_{>0}$ and $i=1,\dots,k-1$. By Equation (6.11) the operators $\mathfrak{q}^i_m$ satisfy the commutation relations

$$[\mathfrak{q}^i_m,\mathfrak{q}^j_n] = m\,\delta_{m,-n}\,C_{ij}\,\mathrm{id} \qquad \text{for} \quad i,j=1,\dots,k-1\ ,\ m,n\in\mathbb{Z}\setminus\{0\}\ ,$$

where $C=(C_{ij})$ is the Cartan matrix of the Dynkin diagram of type A_{k-1}. Let $\mathfrak{L}\subset H^*_T(X_k)_{\mathrm{loc}}$ be the $\mathbb{Z}$-lattice generated by the classes $[D_1]_T,\dots,[D_{k-1}]_T$ with the symmetric bilinear form given by the Cartan matrix C. Then the lattice Heisenberg algebra $\mathfrak{h}_{\mathbb{C}(\varepsilon_1,\varepsilon_2),\mathfrak{L}}$ associated with $\mathfrak{L}$ over $\mathbb{C}(\varepsilon_1,\varepsilon_2)$, which has generators $\mathfrak{q}^i_m$ for $m\in\mathbb{Z}\setminus\{0\}$ and $i=1,\dots,k-1$, is isomorphic to the Heisenberg algebra $\mathfrak{h}_{\mathbb{C}(\varepsilon_1,\varepsilon_2),\mathfrak{Q}}$ of type A_{k-1} over $\mathbb{C}(\varepsilon_1,\varepsilon_2)$ (cf. Example 3.8).

Let

$$E := \sum_{i=0}^{k} a_i\,[D_i]_T \tag{6.19}$$

where $a_i\in\mathbb{C}(\varepsilon_1,\varepsilon_2)$ with $i=0,1,\dots,k$ satisfy the relations

$$2a_j - a_{j-1} - a_{j+1} = 0\ , \quad j=1,\dots,k-1 \qquad \text{and} \qquad a_0\,\varepsilon_2 + a_k\,\varepsilon_1 \neq 0\ . \tag{6.20}$$

The first condition ensures that $\langle [D_i]_T, E\rangle_{\mathbb{H}_1}=0$ for $i=1,\dots,k-1$ while the second condition implies that $\{[D_1]_T,\dots,[D_{k-1}]_T,E\}$ is a $\mathbb{C}(\varepsilon_1,\varepsilon_2)$-linear basis of $H^*_T(X_k)_{\mathrm{loc}}$. By (6.20) one has

$$\kappa := \langle E,E\rangle_{\mathbb{H}_1} = a_0^2\,\beta_1^{-1} - a_0\,a_1 - a_k\,a_{k-1} + a_k^2\,\beta_k\ .$$

From now on we set $a_i=i$ in Equation (6.19) for $i=0,1,\dots,k$, which is consistent with the conditions in Equation (6.20). This implies that $\kappa=k\,\beta$. In the following we normalize the equivariant cohomology class E such that $\langle E,E\rangle_{\mathbb{H}_1}=1$; we denote the normalized class with the same symbol.

Define $\mathfrak{p}_{-m}:=\mathfrak{p}_{-m}(E)$ and $\mathfrak{p}_m:=\mathfrak{p}_m(E)$ for $m\in\mathbb{Z}_{>0}$. Then the operators $\mathfrak{q}^i_m$ and $\mathfrak{p}_m$ satisfy the commutation relations

$$\begin{cases} \left[\mathfrak{q}^i_m,\mathfrak{q}^j_n\right] = m\,\delta_{m,-n}\,C_{ij}\,\mathrm{id} & \text{for} \quad i,j=1,\dots,k-1\ ,\ m,n\in\mathbb{Z}\setminus\{0\}\ ,\\ \left[\mathfrak{q}^i_m,\mathfrak{p}_n\right] = 0 & \text{for} \quad i=1,\dots,k-1\ ,\ m,n\in\mathbb{Z}\setminus\{0\}\ ,\\ \left[\mathfrak{p}_m,\mathfrak{p}_n\right] = m\,\delta_{m,-n}\,\mathrm{id} & \text{for} \quad m,n\in\mathbb{Z}\setminus\{0\}\ . \end{cases}$$

Let $\mathfrak{L}' \subset H^*_T(X_k)_{\mathrm{loc}}$ be the $\mathbb{Z}$-lattice generated by the classes $[D_1]_T, \ldots, [D_{k-1}]_T$ and E. Then the operators $\mathfrak{q}^i_m$ and $\mathfrak{p}_n$ for $m, n \in \mathbb{Z} \setminus \{0\}$ and $1 \leq i \leq k-1$ define the lattice Heisenberg algebra $\mathfrak{h}_{\mathbb{C}(\varepsilon_1,\varepsilon_2),\mathfrak{L}'}$ associated with $\mathfrak{L}'$ over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$. In particular, $\mathfrak{h}_{\mathbb{C}(\varepsilon_1,\varepsilon_2),\mathfrak{L}'}$ is the sum (identifying central elements) of, respectively, the Heisenberg algebra $\mathfrak{h}_{\mathbb{C}(\varepsilon_1,\varepsilon_2),\mathfrak{Q}}$ of type A_{k-1} over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ and the Heisenberg algebra $\mathfrak{h}_{\mathbb{C}(\varepsilon_1,\varepsilon_2)}$ over $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ generated by $\mathfrak{p}_m$ for $m \in \mathbb{Z} \setminus \{0\}$. Since $\{[D_1]_T, \ldots, [D_{k-1}]_T\} \cup \{E\}$ is a $\mathbb{C}(\varepsilon_1, \varepsilon_2)$-linear basis of $H^*_T(X_k)_{\mathrm{loc}}$, we have $\mathfrak{h}_{\mathbb{C}(\varepsilon_1,\varepsilon_2),\mathfrak{L}'} \cong \mathfrak{h}_{\mathbb{H}_1}$. Hence $\mathbb{H}$ is the Fock space of $\mathfrak{h}_{\mathbb{C}(\varepsilon_1,\varepsilon_2),\mathfrak{L}'}$. In what follows we omit the symbol $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ from the notation for the Heisenberg algebras generated over the field $\mathbb{C}(\varepsilon_1, \varepsilon_2)$.

Remark 6.21. By Equation (6.10), for $l = 1, \ldots, k-1$ one has

$$\mathfrak{q}^l_m = -\beta_l\, \mathfrak{p}^l_m + \mathfrak{p}^{l+1}_m \, , \tag{6.22}$$

$$\mathfrak{p}_m = \sqrt{-k\,\varepsilon_1\,\varepsilon_2}\, \sum_{i=1}^{k} \frac{1}{\varepsilon_2^{(i)}}\, \mathfrak{p}^i_m = \frac{1}{\sqrt{\left\langle [X_k], [X_k] \right\rangle_{\mathbb{H}_1}}} \sum_{i=1}^{k} \frac{1}{\varepsilon_2^{(i)}}\, \mathfrak{p}^i_m \, . \tag{6.23}$$

△

6.4 Dominant representation of $\widehat{\mathfrak{gl}}_k$ on $\mathbb{W}_j$

In the following we omit the dependence of the Lie algebras on the field $\mathbb{C}(\varepsilon_1, \varepsilon_2)$ to simplify the presentation.

Proposition 6.24. *Let $j \in \{0, 1, \ldots, k-1\}$. There is an action of $\widehat{\mathfrak{gl}}_k$ on $\mathbb{W}_j$ under which $\mathbb{W}_j$ is the j-th dominant representation of $\widehat{\mathfrak{gl}}_k$ at level one, i.e., the highest weight representation of $\widehat{\mathfrak{gl}}_k$ with fundamental weight $\widehat{\omega}_j$ of type $\widehat{A}_{k-1}$.*

Proof. The vector space $\mathbb{H}$ is an irreducible highest weight representation of the sum (identifying central elements) $\mathfrak{h} \oplus \mathfrak{h}_{\mathfrak{Q}}$ of the Heisenberg algebra $\mathfrak{h}$ and the lattice Heisenberg algebra $\mathfrak{h}_{\mathfrak{Q}}$ of type A_{k-1} generated over the field $\mathbb{C}(\varepsilon_1, \varepsilon_2)$, respectively, by the operators $\mathfrak{p}_m$ and $\mathfrak{q}^i_m$ for $m \in \mathbb{Z} \setminus \{0\}$ and $i = 1, \ldots, k-1$. We apply the Frenkel-Kac construction (Theorem 3.27) to the representation $\mathfrak{h}_{\mathfrak{Q}} \to \mathrm{End}(\mathbb{H})$ to obtain a level one representation

$$\widehat{\mathfrak{sl}}_k \longrightarrow \mathrm{End}\left(\mathbb{H} \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\mathfrak{Q} + \omega_j]\right) .$$

We can extend the representation of $\mathfrak{h}$ from $\mathbb{H}$ to $\mathbb{H} \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\mathfrak{Q} + \omega_j]$ by letting it act as the identity on the group algebra of $\mathfrak{Q} + \omega_j$. Thus we get a level one representation of $\widehat{\mathfrak{gl}}_k$ with

$$\widehat{\mathfrak{gl}}_k \longrightarrow \mathrm{End}\left(\mathbb{H} \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\mathfrak{Q} + \omega_j]\right) .$$

Thanks to the isomorphism Ψ_j introduced in (6.2), this gives a level one representation of $\widehat{\mathfrak{gl}}_k$ with

$$\widehat{\mathfrak{gl}}_k \longrightarrow \mathrm{End}(\mathbb{W}_j) .$$

Since $\mathbb{H}$ is the Fock space of $\mathfrak{h} \oplus \mathfrak{h}_{\mathfrak{Q}}$, the module $\mathbb{H} \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\mathfrak{Q} + \omega_j]$ is isomorphic to the j-th dominant representation $\mathcal{F}_{\mathbb{C}(\varepsilon_1,\varepsilon_2)} \otimes \mathcal{V}(\,\widehat{\omega}_j\,)$ of $\widehat{\mathfrak{gl}}_k$ (cf. Theorem 3.27). Hence to complete the proof it is enough to note that the class $[\emptyset, \boldsymbol{e}_j]$ corresponding to the fixed point $(\mathcal{R}_j, \phi_{\mathcal{R}_j})$ in $\mathcal{M}(\boldsymbol{e}_j, 0, j)$ is sent via Ψ_j to $|0\rangle \otimes [\omega_j]$, which is the highest weight vector of $\mathbb{H} \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\mathfrak{Q} + \omega_j]$. ☐

Remark 6.25. Proposition 6.24 is analogous to a previous result derived for Nakajima quiver varieties (see e.g. [45, Section 10] and [57, Section 5.1]). △

Let us introduce the Virasoro operators of $\widehat{\mathfrak{gl}}_k$ by (cf. Sections 3.1.1 and 3.3.1)

$$L_0 := L_0^{\mathfrak{h}} + L_0^{\widehat{\mathfrak{sl}}_k} = \sum_{m=1}^{\infty} \mathfrak{p}_{-m}\,\mathfrak{p}_m + \sum_{i=1}^{k-1}\sum_{m=1}^{\infty} \mathfrak{q}_{-m}^{\eta_i}\,\mathfrak{q}_m^{\eta_i} + \frac{1}{2}\sum_{i=1}^{k-1}\big(\mathfrak{q}_0^{\eta_i}\big)^2 ,$$

$$L_n := L_n^{\mathfrak{h}} + L_n^{\widehat{\mathfrak{sl}}_k} = \frac{1}{2}\sum_{m\in\mathbb{Z}} \mathfrak{p}_{-m}\,\mathfrak{p}_{m+n} + \frac{1}{2}\sum_{i=1}^{k-1}\sum_{m\in\mathbb{Z}} \mathfrak{q}_{-m}^{\eta_i}\,\mathfrak{q}_{m+n}^{\eta_i} \qquad \text{for} \quad n\neq 0 ,$$

where $\{\eta_i\}_{i=1}^{k-1}$ is an orthonormal basis of the vector space $\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{R}$ and we set $\mathfrak{p}_0 := 0$. Note that $\{\eta_i\}_{i=1}^{k-1}\cup\{E\}$ is an orthonormal basis of the vector space $\mathbb{C}(\varepsilon_1,\varepsilon_2)^k \simeq H_T^*(X_k)_{\mathrm{loc}}$, so after an orthonormal change of basis and a suitable normalization one can rewrite the operators L_n in the form

$$L_0 = \sum_{l=1}^{k}\beta_l \sum_{m=1}^{\infty} \mathfrak{p}_{-m}^{l}\,\mathfrak{p}_m^{l} + \frac{1}{2}\sum_{i=1}^{k-1}\big(\mathfrak{q}_0^{\eta_i}\big)^2 ,$$

$$L_n = \frac{1}{2}\sum_{l=1}^{k}\beta_l \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0,-n}} \mathfrak{p}_{-m}^{l}\,\mathfrak{p}_{m+n}^{l} + \sum_{i=1}^{k-1}\mathfrak{q}_0^{\eta_i}\,\mathfrak{q}_n^{\eta_i} \qquad \text{for} \quad n\neq 0 .$$

Proposition 6.26. *Let $j\in\{0,1,\dots,k-1\}$, $\vec{u}\in\mathfrak{U}_j$ and $n\in\mathbb{N}$. Then*

$$L_0\big|_{\mathbb{W}_{\vec{u},n,j}} = \big(n + \tfrac{1}{2}\,\vec{v}\cdot C\vec{v}\big)\ \mathrm{id}_{\mathbb{W}_{\vec{u},n,j}} ,$$

where $\vec{v} := C^{-1}\vec{u}$.

Proof. By Equation (4.21), we have $\sum_{m=1}^{\infty}\mathfrak{p}_{-m}^{l}\,\mathfrak{p}_m^{l}\triangleright[\vec{Y}] = \beta_l^{-1}\,|Y^l|\,[\vec{Y}]$ for $[\vec{Y}] = [(\emptyset,\dots,Y^l,\dots,\emptyset)]$ in $\mathbb{H}_{U_l}$ and $l\in\{1,\dots,k\}$. Then by using the isomorphism Ω introduced in (6.8) and the isomorphism Ψ_j introduced in (6.2) we get

$$\Big(\sum_{l=1}^{k}\beta_l\sum_{m=1}^{\infty}\mathfrak{p}_{-m}^{l}\,\mathfrak{p}_m^{l}\Big)\Big|_{\mathbb{W}_{\vec{u},n,j}} = n\ \mathrm{id}_{\mathbb{W}_{\vec{u},n,j}} .$$

On the other hand, since $\{\eta_i\}_{i=1}^{k-1}$ is an orthonormal basis of $\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{R}$ we get

$$\Big(\sum_{i=1}^{k-1}\big(\mathfrak{q}_0^{\eta_i}\big)^2\Big)\Big|_{\mathbb{W}_{\vec{u},n,j}} = \Big(\sum_{i=1}^{k-1}\Big\langle \eta_i\,,\,\sum_{s=1}^{k-1}v_s\,\gamma_s\Big\rangle_{\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{R}}^2\Big)\ \mathrm{id}_{\mathbb{W}_{\vec{u},n,j}} = \big(\vec{v}\cdot C\vec{v}\big)\ \mathrm{id}_{\mathbb{W}_{\vec{u},n,j}} ,$$

where $\vec{v} := C^{-1}\vec{u}$. □

Since

$$L_0\triangleright[\emptyset,\vec{u}\,] = \tfrac{1}{2}\,\vec{v}\cdot C\vec{v}\,[\emptyset,\vec{u}\,] \qquad \text{and} \qquad L_n\triangleright[\emptyset,\vec{u}\,] = 0 \quad \text{for} \quad n>0 ,$$

we have the following result.

Corollary 6.27. *$\mathbb{W}_{\vec{u},j}$ is a highest weight representation of the Virasoro algebra $\mathfrak{Vir}$ associated with $\widehat{\mathfrak{gl}}_k$, which is generated by the operators L_n and $\mathfrak{c}$ with highest weight vector $[\emptyset, \vec{u}\,]$ and conformal dimension*

$$\Delta_{\vec{u}} := \tfrac{1}{2}\,\vec{v}\cdot C\vec{v}\ .$$

Moreover, the energy eigenspace decomposition of the representation $\mathbb{W}_{\vec{u},j}$ is given by (5.17).

Proposition 6.28. *Let $j \in \{0, 1, \ldots, k-1\}$. The weight decomposition of $\mathbb{W}_j$ as a $\widehat{\mathfrak{gl}}_k$-module is given by*

$$\mathbb{W}_j = \bigoplus_{\vec{u}\in\mathfrak{U}_j} \mathbb{W}_{\vec{u},j}\ .$$

Proof. For $j = 0, 1, \ldots, k-1$ and for any element $A \otimes [\gamma_{\vec{u}} + \omega_j] \in \mathbb{H} \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\mathfrak{Q} + \omega_j]$ with $\vec{u} \in \mathfrak{U}_j$, we have

$$\begin{aligned} h_0 \triangleright (A \otimes [\gamma_{\vec{u}} + \omega_j]) &= \Big(1 - \sum_{l=1}^{k-1} u_l\Big)\, (A \otimes [\gamma_{\vec{u}} + \omega_j])\ , \\ h_i \triangleright (A \otimes [\gamma_{\vec{u}} + \omega_j]) &= u_i\, (A \otimes [\gamma_{\vec{u}} + \omega_j]) \qquad \text{for} \quad i = 1, \ldots, k-1\ . \end{aligned}$$

Under the $\widehat{\mathfrak{gl}}_k$-action, $\mathbb{W}_{\vec{u},j}$ decomposes as

$$\mathbb{W}_{\vec{u},j} \simeq \mathcal{F}_{\mathbb{C}(\varepsilon_1,\varepsilon_2)} \otimes \mathcal{V}(\widehat{\omega}_j)_{\gamma_{\vec{u}}}\ ,$$

where $\mathcal{V}(\widehat{\omega}_j)_\lambda := \{w \in \mathcal{V}(\widehat{\omega}_j) \,|\, h_i \triangleright w = (\widehat{\omega}_j + \lambda)(h_i)\, w\}$ for any weight λ. The assertion now follows by showing that for a weight λ, the vector space $\mathcal{V}(\widehat{\omega}_j)_\lambda$ is nonzero if and only if $\lambda = \gamma_{\vec{u}}$ for some $\vec{u} \in \mathfrak{U}_j$. For this, let $\vec{\xi}^{\,\vec{v}} := \prod_{i=1}^{k-1} \xi_i^{v_i}$ for $\xi_i \in \mathbb{C}^*$ with $|\xi_i| < 1$ and any vector $\vec{v} = (v_1, \ldots, v_{k-1})$, and set $\vec{h} := (h_1, \ldots, h_{k-1})$. Then it is enough to note that the trace of the operator $\mathsf{q}^{L_0}\, \vec{\xi}^{\,C^{-1}\vec{h}}$ is given by

$$\mathrm{Tr}_{\mathbb{W}_j}\ \mathsf{q}^{L_0}\, \vec{\xi}^{\,C^{-1}\vec{h}} = \sum_{\vec{u}\in\mathfrak{U}_j}\ \sum_{n=0}^{\infty} \mathsf{q}^{n+\frac{1}{2}\,\vec{v}\cdot C\vec{v}}\ \vec{\xi}^{\,\vec{v}}$$

and the right-hand side is exactly the character of the j-th dominant representation of $\widehat{\mathfrak{gl}}_k$ by [14, Section 5.3]. ◻

6.4.1 Whittaker vectors

Consider now the completed total equivariant cohomology

$$\widehat{\mathbb{W}}_j := \prod_{\vec{u}\in\mathfrak{U}_j}\ \prod_{n\geq 0} \mathbb{W}_{\vec{u},n,j}\ .$$

We can extend the isomorphism (6.2) to

$$\widehat{\Psi}_j\ :\ \widehat{\mathbb{W}}_j \ \xrightarrow{\ \sim\ }\ \widehat{\mathbb{H}} \otimes \Big(\prod_{\gamma_{\vec{u}}\in\mathfrak{Q}} \mathbb{C}(\varepsilon_1, \varepsilon_2)\, (\gamma_{\vec{u}} + \omega_j) \Big)\ ,$$

where $\widehat{\mathbb{H}} := \prod_{n\geq 0} H_T^*(\mathrm{Hilb}^n(X_k))_{\mathrm{loc}}$ is the completed total equivariant cohomology of the Hilbert schemes of points on X_k. In the following we drop the explicit symbols $\widehat{\Psi}_j$ from the notation in order to simplify the presentation, and we denote

$$|\omega_j\rangle := \sum_{\vec{u}\in\mathfrak{U}_j} [\emptyset, \vec{u}\,] \ .$$

Proposition 6.29. *Fix $\vec{\eta} \in \mathbb{C}(\varepsilon_1, \varepsilon_2)^k$. In the completed total equivariant cohomology $\widehat{\mathbb{W}}_j$ the vector*

$$G_j(\vec{\eta}\,) := \exp\Big(\sum_{i=1}^k \eta_i\ \mathfrak{p}_{-1}^i\Big)|\omega_j\rangle$$

is a Whittaker vector of type $\chi_{\vec{\eta}}$, where $\chi_{\vec{\eta}}\colon \mathcal{U}(\mathfrak{h}^+ \oplus \mathfrak{h}_{\mathfrak{Q}}^+) \to \mathbb{C}(\varepsilon_1, \varepsilon_2)$ is defined by

$$\chi_{\vec{\eta}}(\mathfrak{q}_1^i) = \eta_{i+1}\,\beta_{i+1}^{-1} - \eta_i \qquad \text{and} \qquad \chi_{\vec{\eta}}(\mathfrak{q}_m^i) = 0 \quad \text{for} \quad m > 1\ ,\ i = 1, \ldots, k-1\ ,$$

$$\chi_{\vec{\eta}}(\mathfrak{p}_1) = \frac{1}{\sqrt{\big\langle [X_k], [X_k]\big\rangle_{\mathbb{H}_1}}}\ \sum_{i=1}^k \frac{\eta_i}{\varepsilon_1^{(i)}} \qquad \text{and} \qquad \chi_{\vec{\eta}}(\mathfrak{p}_m) = 0 \quad \text{for} \quad m > 1\ .$$

Proof. Let $\widehat{\mathbb{H}}_{U_i} := \prod_{n\geq 0} H_T^*(\mathrm{Hilb}^n(U_i))_{\mathrm{loc}}$ be the completed total equivariant cohomology of the Hilbert scheme $\mathrm{Hilb}^n(U_i)$ for $i = 1, \ldots, k$, and define $G(\eta_i) := \exp\big(\eta_i\ \mathfrak{p}_{-1}^i\big)\,|0\rangle \in \widehat{\mathbb{H}}_{U_i}$. By using Theorem 6.18 and the completed versions of the isomorphisms (6.8) and (6.2), we can rewrite the vector $G_j(\vec{\eta}\,)$ as

$$G_j(\vec{\eta}\,) = G(\eta_1) \otimes \cdots \otimes G(\eta_k)\ \otimes \sum_{\vec{u}\in\mathfrak{U}_j} (\gamma_{\vec{u}} + \omega_j)\ .$$

By Proposition 4.13, $G(\eta_i)$ for $i = 1, \ldots, k$ is a Whittaker vector for the Heisenberg algebra $\mathfrak{h}_i$ of type χ_i, where

$$\chi_i(\mathfrak{p}_1^i) := \eta_i\,\beta_i^{-1} \qquad \text{and} \qquad \chi_i(\mathfrak{p}_m^i) := 0 \quad \text{for} \quad m > 1\ .$$

Again by Theorem 6.18, each $\mathfrak{h}_i$ acts trivially on $\mathbb{H}_{U_l}$ for $l \neq i$ and it is easy to see that $G_j(\vec{\eta}\,)$ is a Whittaker vector for the Heisenberg algebra $\mathfrak{h}_{\mathbb{H}_1}$ of type $\tilde{\chi}_{\vec{\eta}}$, where $\tilde{\chi}_{\vec{\eta}}\colon \mathcal{U}\big(\mathfrak{h}_{\mathbb{H}_1}^+\big) \to \mathbb{C}(\varepsilon_1, \varepsilon_2)$ is defined by $\tilde{\chi}_{\vec{\eta}}(\mathfrak{p}_m^i) := \chi_i(\mathfrak{p}_m^i)$ for $i = 1, \ldots, k$ and $m \in \mathbb{Z} \setminus \{0\}$. Then by Remark 6.21, $G_j(\vec{\eta}\,)$ is a Whittaker vector for $\widehat{\mathfrak{gl}}_k$ of type $\chi_{\vec{\eta}}$ with $\chi_{\vec{\eta}}\colon \mathcal{U}(\mathfrak{h}^+ \oplus \mathfrak{h}_{\mathfrak{Q}}^+) \to \mathbb{C}(\varepsilon_1, \varepsilon_2)$ defined for every $m > 0$ by

$$\chi_{\vec{\eta}}(\mathfrak{q}_m^i) := \tilde{\chi}_{\vec{\eta}}(\mathfrak{p}_m^{i+1}) - \beta_i\,\tilde{\chi}_{\vec{\eta}}(\mathfrak{p}_m^i) = \delta_{m,1}\,\big(\eta_{i+1}\,\beta_{i+1}^{-1} - \eta_i\big)\ ,$$

$$\chi_{\vec{\eta}}(\mathfrak{p}_m) := \sqrt{-k\,\varepsilon_1\,\varepsilon_2}\ \sum_{i=1}^k \frac{1}{\varepsilon_2^{(i)}}\,\tilde{\chi}_{\vec{\eta}}(\mathfrak{p}_m^i) = \delta_{m,1}\,\sqrt{-k\,\varepsilon_1\,\varepsilon_2}\ \sum_{i=1}^k \frac{\eta_i}{\varepsilon_1^{(i)}}\ .$$

□

7 Chiral vertex operators for $\widehat{\mathfrak{gl}}_k$

7.1 Ext-bundles and bifundamental hypermultiplets

In this section we construct and study the natural generalizations of the Ext vertex operators from Section 4.3 for the moduli spaces $\mathcal{M}(\vec{u}, n, j)$.

For $n \in \mathbb{N}$, $j \in \{0, 1, \ldots, k-1\}$ and $\vec{u} \in \mathfrak{U}_j$, let $\boldsymbol{\mathcal{E}}_{\vec{u},n,j}$ denote the universal sheaf on $\mathcal{M}(\vec{u}, n, j) \times \mathscr{X}_k$. Define

$$\boldsymbol{\mathcal{E}}_i := p_{i3}^*\big(\boldsymbol{\mathcal{E}}_{\vec{u}_i,n_i,j_i}\big) \ \in \ K\big(\mathcal{M}(\vec{u}_1, n_1; j_1) \times \mathcal{M}(\vec{u}_2, n_2; j_2) \times \mathscr{X}_k\big) \qquad \text{for} \quad i = 1, 2 \ ,$$

where p_{ij} is the projection of $\mathcal{M}(\vec{u}_1, n_1; j_1) \times \mathcal{M}(\vec{u}_2, n_2; j_2) \times \mathscr{X}_k$ onto the product of the i-th and j-th factors. Denote by p_3 the projection of the same product onto $\mathscr{X}_k$.

Let $T_\mu = \mathbb{C}^*$ and $H^*_{T_\mu}(\mathrm{pt}; \mathbb{C}) = \mathbb{C}[\mu]$. Denote by $\mathcal{O}_{\mathscr{X}_k}(\mu)$ the trivial line bundle on $\mathscr{X}_k$ on which T_μ acts by scaling the fibers.

Definition 7.1 ([14, Definition 4.17])**.** The *Carlsson-Okounkov bundle* is the element

$$\boldsymbol{E}_\mu^{\vec{u}_1,n_1,j_1;\vec{u}_2,n_2,j_2} := p_{12*}\big(-\boldsymbol{\mathcal{E}}_1^\vee \cdot \boldsymbol{\mathcal{E}}_2 \cdot p_3^*(\mathcal{O}_{\mathscr{X}_k}(\mu) \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))\big)$$

in the K-theory $K\big(\mathcal{M}(\vec{u}_1, n_1; j_1) \times \mathcal{M}(\vec{u}_2, n_2; j_2)\big)$. ⊘

By [14, Section 4.5], the fibre of $\boldsymbol{E}_\mu^{\vec{u}_1,n_1,j_1;\vec{u}_2,n_2,j_2}$ over $\big([(\mathcal{E}, \phi_\mathcal{E})]\ ,\ [(\mathcal{E}', \phi_{\mathcal{E}'})]\big)$ in $\mathcal{M}(\vec{u}_1, n_1; j_1) \times \mathcal{M}(\vec{u}_2, n_2; j_2)$ is given by

$$\boldsymbol{E}_\mu^{\vec{u}_1,n_1,j_1;\vec{u}_2,n_2,j_2}\big|_{([(\mathcal{E},\phi_\mathcal{E})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])} = \mathrm{Ext}^1\left(\mathcal{E}, \mathcal{E}' \otimes \mathcal{O}_{\mathscr{X}_k}(\mu) \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\right) .$$

One can compute the dimension of this vector space by a straightforward generalization of the dimension computations of [14, Appendix A] to get the rank

$$\mathrm{rk}\left(\boldsymbol{E}_\mu^{\vec{u}_1,n_1,j_1;\vec{u}_2,n_2,j_2}\right) = n_1 + n_2 + \tfrac{1}{2}\,\vec{v}_{21} \cdot C\vec{v}_{21} - \tfrac{1}{2k}\, j_{21}\,(k - j_{21}) \ ,$$

where $\vec{v}_{21} := C^{-1}(\vec{u}_2 - \vec{u}_1)$ and $j_{21} \in \{0, 1, \ldots, k-1\}$ is the equivalence class modulo k of $j_2 - j_1$.

Let $\mathbb{W} := \bigoplus_{j=0}^{k-1} \mathbb{W}_j$ endowed with the nondegenerate $\mathbb{C}(\varepsilon_1, \varepsilon_2)$-valued bilinear form $\langle -, - \rangle_\mathbb{W}$ induced by the symmetric bilinear forms $\langle -, - \rangle_{\mathbb{W}_j}$. Define the operator $\mathrm{V}_\mu(\vec{x}, z) \in \mathrm{End}(\mathbb{W})[[z^{\pm 1}, x_1^{\pm 1}, \ldots, x_{k-1}^{\pm 1}]]$ by its matrix elements

$$\begin{aligned}(-1)^{n_2}\,\big\langle \mathrm{V}_\mu(\vec{x}, z)A_1\,,\, A_2\big\rangle_\mathbb{W} &:= z^{n_2-n_1+\Delta_{\vec{u}_2}-\Delta_{\vec{u}_1}}\ \vec{x}^{\,\vec{v}_{21}} \\ &\quad\times \int_{\mathcal{M}(\vec{u}_1,n_1;j_1)\times\mathcal{M}(\vec{u}_2,n_2;j_2)} \mathrm{eu}_T\left(\boldsymbol{E}_\mu^{\vec{u}_1,n_1,j_1;\vec{u}_2,n_2,j_2}\right) \cup p_1^*(A_1) \cup p_2^*(A_2)\ , \end{aligned} \tag{7.2}$$

where $A_i \in H_T^*(\mathcal{M}(\vec{u}_i, n_i; j_i))_{\mathrm{loc}}$ and p_i is the projection from $\mathcal{M}(\vec{u}_1, n_1; j_1) \times \mathcal{M}(\vec{u}_2, n_2; j_2)$ to the i-th factor for $i = 1, 2$. The extra isospin parameters $\vec{x} := (x_1, \ldots, x_{k-1})$ weigh the $\widehat{\mathfrak{sl}}_k$-action, and we abbreviated $\vec{x}^{\,\vec{v}} := \prod_{i=1}^{k-1} x_i^{v_i}$ for a vector $\vec{v} = (v_1, \ldots, v_{k-1})$. By the computations of [14, Section 4.7], the matrix elements (7.2) in the fixed point basis are given by

$$\begin{aligned}&\big\langle \mathrm{V}_\mu(\vec{x}, z)[\vec{Y}_1, \vec{u}_1\,]\,,\, [\vec{Y}_2, \vec{u}_2\,]\big\rangle_\mathbb{W} \\ &= (-1)^{|\vec{Y}_2|}\, z^{|\vec{Y}_2|-|\vec{Y}_1|+\Delta_{\vec{u}_2}-\Delta_{\vec{u}_1}}\ \vec{x}^{\,\vec{v}_{21}}\ \ \mathrm{eu}_T\left(\boldsymbol{E}_\mu^{\vec{u}_1,n_1,j_1;\vec{u}_2,n_2,j_2}\big|_{([(\mathcal{E}_1,\phi_{\mathcal{E}_1})]\,,\,[(\mathcal{E}_2,\phi_{\mathcal{E}_2})])}\right)\end{aligned}$$

$$= (-1)^{|\vec{Y}_2|}\, z^{|\vec{Y}_2|-|\vec{Y}_1|+\Delta_{\vec{u}_2}-\Delta_{\vec{u}_1}}\, \vec{x}^{\,\vec{v}_{21}} \prod_{i=1}^{k} m_{Y_1^i,Y_2^i}\big(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \mu - (\vec{v}_{21})_i\, \varepsilon_1^{(i)} - (\vec{v}_{21})_{i-1}\, \varepsilon_2^{(i)}\big)$$
$$\times \prod_{n=1}^{k-1} \ell^{(n)}_{\vec{v}_{21}}\big(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, \mu\big)\ , \qquad (7.3)$$

where $[(\mathcal{E}_1, \phi_{\mathcal{E}_1})]$ and $[(\mathcal{E}_2, \phi_{\mathcal{E}_2})]$ are the T-fixed points corresponding respectively to the combinatorial data $(\vec{Y}_1, \vec{u}_1)$ and $(\vec{Y}_2, \vec{u}_2)$, and we use the convention $(\vec{v}_{21})_0 = (\vec{v}_{21})_k = 0$. Here m_{Y_1,Y_2} is defined in (4.17), while $\ell^{(n)}_{\vec{v}}$ is the *edge contribution* defined in Appendix B. This factorized expression for the matrix elements represents the contribution of the $U(1) \times U(1)$ bifundamental hypermultiplet for $\mathcal{N} = 2$ quiver gauge theories on the ALE space X_k.

7.2 Vertex operators and primary fields

In this subsection we factorize the operators $\mathrm{V}_\mu(\vec{x}, z)$ defined in (7.2) under the decomposition $\widehat{\mathfrak{gl}}_k = \mathfrak{h} \oplus \widehat{\mathfrak{sl}}_k$ as tensor products of generalized bosonic exponentials associated to the Heisenberg algebra $\mathfrak{h}$ from Definition 3.3 with primary fields of the Virasoro algebra associated to the affine Lie algebra $\widehat{\mathfrak{sl}}_k$ from Section 3.3.1.

For $l = 1, 2$ fix $j_l \in \{0, 1, \ldots, k-1\}$ and respectively $\vec{u}_l \in \mathfrak{U}_{j_l}$. Set

$$\gamma_{21} := \sum_{i=1}^{k-1} (\vec{v}_{21})_i\, \gamma_i = \psi_{j_2}(\vec{u}_2) - \psi_{j_1}(\vec{u}_1) \ \in\ \mathfrak{Q} \otimes_{\mathbb{Z}} \mathbb{Q}\ .$$

Note that $\gamma_{21} = \gamma_{\vec{u}_2} - \gamma_{\vec{u}_1} + \omega_{j_2} - \omega_{j_1}$. We define the maps

$$\begin{aligned} \exp\big(\gamma_{21}\big) :\ \mathbb{H} \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\mathfrak{Q} + \omega_{j_1}] \ &\longrightarrow\ \mathbb{H} \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\mathfrak{Q} + \omega_{j_2}]\ , \\ v \otimes [\beta + \omega_{j_1}] \ &\longmapsto\ v \otimes [\beta + \gamma_{\vec{u}_2} - \gamma_{\vec{u}_1} + \omega_{j_2}]\ , \end{aligned}$$

and $\exp\big(\log z\, \mathfrak{c} + \gamma_{21}\big) :\ \mathbb{H} \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\mathfrak{Q} + \omega_{j_1}] \ \to\ \mathbb{H} \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\mathfrak{Q} + \omega_{j_2}]$, given by

$$\exp\big(\log z\, \mathfrak{c} + \gamma_{21}\big) \triangleright (v \otimes [\beta + \omega_{j_1}]) := z^{\frac{1}{2}\langle \gamma_{21}, \gamma_{21}\rangle_{\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{Q}} + \langle \gamma_{21}, \beta + \omega_{j_1}\rangle_{\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{Q}}}\, (v \otimes [\beta + \gamma_{\vec{u}_2} - \gamma_{\vec{u}_1} + \omega_{j_2}])\ .$$

Note that the operator $\exp\big(\log z\ \mathfrak{c} - \gamma_{21}\big)\ \exp\big(\gamma_{21}\big) \in \mathrm{End}(\mathbb{H} \otimes \mathbb{C}(\varepsilon_1, \varepsilon_2)[\mathfrak{Q} + \omega_{j_1}])[[z, z^{-1}]]$ is given by

$$\begin{aligned} &\exp\big(\log z\, \mathfrak{c} - \gamma_{21}\big)\ \exp\big(\gamma_{21}\big) \triangleright (v \otimes [\beta + \omega_{j_1}]) \\ &\qquad = z^{-\frac{1}{2}\langle \gamma_{21}, \gamma_{21}\rangle_{\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{Q}} - \langle \gamma_{21}, \beta + \omega_{j_1}\rangle_{\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{Q}}}\, (v \otimes [\beta + \omega_{j_1}])\ . \qquad (7.4) \end{aligned}$$

In the following we shall suppress the explicit isomorphism Ψ from Theorem 6.1 in our formulas in order to simplify the presentation.

We will now rewrite the operator $\mathrm{V}_\mu(\vec{x}, z)$ in terms of chiral vertex operators in $\mathrm{Hom}(\mathbb{W}_{\vec{u}_1;j_1}, \mathbb{W}_{\vec{u}_2;j_2})[[z^{\pm 1}, x_1^{\pm 1}, \ldots, x_{k-1}^{\pm 1}]]$ between two highest weight representations of the Virasoro algebra associated with $\widehat{\mathfrak{sl}}_k$. For this, let us define the vertex operator $\bar{\mathrm{V}}_\mu(\vec{v}_{21}, \vec{x}, z)$ of $\mathrm{Hom}(\mathbb{W}_{j_1}, \mathbb{W}_{j_2})[[z^{\pm 1}, x_1^{\pm 1}, \ldots, x_{k-1}^{\pm 1}]]$ by

$$\bar{\mathrm{V}}_\mu(\vec{v}_{21}, \vec{x}, z) := \vec{x}^{\,\vec{v}_{21}} \prod_{l=1}^{k-1} \ell^{(l)}_{\vec{v}_{21}}\big(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \mu\big)\ \mathrm{V}^{\gamma_{21}}_{1,-1}(z)\ \exp\big(\log z\, \mathfrak{c} + \gamma_{21}\big)\ , \qquad (7.5)$$

where $\mathrm{V}^{\gamma_{21}}_{1,-1}(z)$ is the normal-ordered bosonic exponential associated with the Heisenberg algebra $\mathfrak{h}_{\mathfrak{Q}}$.

Theorem 7.6. *Under the decomposition $\widehat{\mathfrak{gl}}_k = \mathfrak{h} \oplus \widehat{\mathfrak{sl}}_k$, the operator $\mathrm{V}_\mu(\vec{x}, z)$ is given in terms of products of vertex operators as*

$$
\begin{aligned}
\mathrm{V}_\mu(\vec{x}, z) &= \mathrm{V}_{-\frac{\mu}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}}, \frac{\mu+\varepsilon_1+\varepsilon_2}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}}}(z) \\
&\quad \otimes \sum_{j_1, j_2=0}^{k-1} \ \sum_{\vec{u}_1 \in \mathfrak{U}_{j_1}, \vec{u}_2 \in \mathfrak{U}_{j_2}} \bar{\mathrm{V}}_\mu(\vec{v}_{21}, \vec{x}, z)\, z^{\Delta_{\vec{u}_2} - \Delta_{\vec{u}_1}} \ \exp\big(\log z\, \mathfrak{c} - \gamma_{21}\big) \ \exp\big(\gamma_{21}\big)\big|_{\mathbb{W}_{\vec{u}_1, j_1}} \ ,
\end{aligned}
$$

where $\bar{\mathrm{V}}_\mu(\vec{v}_{21}, \vec{x}, z)$ is a primary field of the Virasoro algebra generated by $L_n^{\widehat{\mathfrak{sl}}_k}$ and $\mathfrak{c}$ with conformal dimension $\Delta_{\vec{u}_2 - \vec{u}_1} = \frac{1}{2}\, \vec{v}_{21} \cdot C \vec{v}_{21}$, i.e., for any $n \in \mathbb{Z}$ we have

$$
\big[L_n^{\widehat{\mathfrak{sl}}_k}, \bar{\mathrm{V}}_\mu(\vec{v}_{21}, \vec{x}, z)\big] = z^n \big(z\, \partial_z + \tfrac{1}{2}\, \vec{v}_{21} \cdot C \vec{v}_{21}\, n\big) \bar{\mathrm{V}}_\mu(\vec{v}_{21}, \vec{x}, z) \ . \tag{7.7}
$$

Proof. By using the isomorphism Ψ and Equation (4.16) we get

$$
\begin{aligned}
\mathrm{V}_\mu(\vec{x}, z) &= \Psi^{-1} \circ \Big(\sum_{j_1, j_2=0}^{k-1} \ \sum_{\vec{u}_1 \in \mathfrak{U}_{j_1}, \vec{u}_2 \in \mathfrak{U}_{j_2}} z^{\Delta_{\vec{u}_2} - \Delta_{\vec{u}_1}}\, \vec{x}^{\,\vec{v}_{21}} \prod_{n=1}^{k-1} \ell^{(n)}_{\vec{v}_{21}}\big(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, \mu\big) \\
&\quad \times \ :\prod_{i=1}^{k} \mathrm{V}\left(\mathcal{O}_{U_i}\big(\mu - (\vec{v}_{21})_i\, \varepsilon_1^{(i)} - (\vec{v}_{21})_{i-1}\, \varepsilon_2^{(i)}\big)\,,\, z\right) : \ \otimes \big(\psi_{j_2}(\vec{u}_2) \otimes \psi_{j_1}(\vec{u}_1)^*\big) \Big) \circ \Psi
\end{aligned}
$$

where $\mathcal{O}_{U_i}(\mu)$ is the trivial line bundle on $U_i \simeq \mathbb{C}^2$ with an action of T_μ which rescales the fibers, and $\psi_{j_1}(\vec{u}_1)^*$ denotes the dual vector to $\psi_{j_1}(\vec{u}_1)$ in the dual vector space $\mathbb{C}(\varepsilon_1, \varepsilon_2)[\mathfrak{Q} + \omega_{j_1}]^*$. By Theorem 4.18 we get an expression determined by the operators $\mathfrak{p}^i_m$ for $m \in \mathbb{Z} \setminus \{0\}$ and $i = 1, \dots, k$ as

$$
\begin{aligned}
&\mathrm{V}\left(\mathcal{O}_{U_i}\big(\mu - (\vec{v}_{21})_i\, \varepsilon_1^{(i)} - (\vec{v}_{21})_{i-1}\, \varepsilon_2^{(i)}\big)\,,\, z\right) \\
&\qquad = \exp\Big(- \frac{\mu - (\vec{v}_{21})_i\, \varepsilon_1^{(i)} - (\vec{v}_{21})_{i-1}\, \varepsilon_2^{(i)}}{\varepsilon_2^{(i)}} \sum_{m=1}^{\infty} \frac{z^m}{m}\, \mathfrak{p}^i_{-m} \Big) \\
&\qquad\qquad \times \ \exp\Big(\frac{\mu + \varepsilon_1 + \varepsilon_2 - (\vec{v}_{21})_i\, \varepsilon_1^{(i)} - (\vec{v}_{21})_{i-1}\, \varepsilon_2^{(i)}}{\varepsilon_2^{(i)}} \sum_{m=1}^{\infty} \frac{z^{-m}}{m}\, \mathfrak{p}^i_m \Big) \ .
\end{aligned}
$$

By using Equations (6.22) and (6.23), we can rewrite $:\prod_{i=1}^k \mathrm{V}\left(\mathcal{O}_{U_i}\big(\mu - (\vec{v}_{21})_i\, \varepsilon_1^{(i)} - (\vec{v}_{21})_{i-1}\, \varepsilon_2^{(i)}\big)\,,\, z\right):$ in terms of Heisenberg operators $\mathfrak{q}^l_m$ and $\mathfrak{p}_m$, for $l = 1, \dots, k-1$ and $m \in \mathbb{Z} \setminus \{0\}$, and the first assertion now follows. The proof of Equation (7.7) is somewhat lengthy and can be found in Appendix A. □

Remark 7.8. In the following we will denote by $\mathrm{V}^{j_1, j_2}_\mu(\vec{x}, z)$ the restriction of the vertex operator $\mathrm{V}_\mu(\vec{x}, z)$ to $\mathrm{Hom}(\mathbb{W}_{j_1}, \mathbb{W}_{j_2})[[z^{\pm 1}, x_1^{\pm 1}, \dots, x_{k-1}^{\pm 1}]]$. △

7.3 Integrals of motion

Let $\boldsymbol{V}^{\vec{u},n,j}$ be the pushforward of $\boldsymbol{E}_0^{\vec{u},n,j;\vec{u},n,j}$ with respect to the projection of the product $\mathcal{M}(\vec{u},n,j)\times\mathcal{M}(\vec{u},n,j)$ to the second factor. It is a T-equivariant vector bundle on $\mathcal{M}(\vec{u},n,j)$ of rank $n+\frac{1}{2}\,(\vec{v}\cdot C\vec{v}-\frac{1}{k}\,j\,(k-j))$, which we shall call the natural bundle over $\mathcal{M}(\vec{u},n,j)$. The T-equivariant Chern character of $\boldsymbol{V}^{\vec{u},n,j}$ at a fixed point $[(\mathcal{E},\phi_{\mathcal{E}})]$ with combinatorial datum $(\vec{Y},\vec{u}\,)$ is given by (cf. [14, Section 4.7])

$$\mathrm{ch}_T\big(\boldsymbol{V}^{\vec{u},n,j}\big|_{[(\mathcal{E},\phi_{\mathcal{E}})]}\big)=\sum_{i=1}^{k}\sum_{s\in Y^i}\mathrm{e}^{\,-(v_i+L'_{Y^i}(s))\,\varepsilon_1^{(i)}-(v_{i-1}+A'_{Y^i}(s))\,\varepsilon_2^{(i)}}+\sum_{n=1}^{k-1}L_{\vec{v}}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)}\big)\ ,$$

where the edge contributions $L_{\vec{v}}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)}\big)$ are defined in Appendix B.

Let us denote by $\boldsymbol{V}^j$ the natural bundle over $\coprod_{\vec{u}\in\mathfrak{U}_j}\coprod_{n\geq 0}\mathcal{M}(\vec{u},n,j)$, and consider the operators of multiplication by $\boldsymbol{I}_1:=\mathrm{rk}\big(\boldsymbol{V}^j\big)$ and $\boldsymbol{I}_p:=(\mathrm{c}_{p-1})_T\big(\boldsymbol{V}^j\big)$ for $p\geq 2$ on $\prod_{\vec{u}\in\mathfrak{U}_j}\prod_{n\geq 0}\mathbb{W}_{\vec{u},n,j}$. For example, one has

$$\begin{aligned}\boldsymbol{I}_1\triangleright[\vec{Y},\vec{u}\,]&=\big(|\vec{Y}|+\tfrac{1}{2}\,\vec{v}\cdot C\vec{v}\,\big)\,[\vec{Y},\vec{u}\,]-\tfrac{1}{2k}\,j\,(k-j)\,[\vec{Y},\vec{u}\,]\ ,\\ \boldsymbol{I}_2\triangleright[\vec{Y},\vec{u}\,]&=\ -\sum_{i=1}^{k}\sum_{s\in Y^i}\Big(\big(v_i+L'_{Y^i}(s)\big)\,\varepsilon_1^{(i)}+\big(v_{i-1}+A'_{Y^i}(s)\big)\,\varepsilon_2^{(i)}\Big)\,[\vec{Y},\vec{u}\,]\\ &\qquad+\sum_{n=1}^{k-1}\ell_{\vec{v}}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)}\big)_{[1]}\,[\vec{Y},\vec{u}\,]\ ,\end{aligned}$$

where the quantities $\ell_{\vec{v}}^{(n)}(\varepsilon_1^{(n)},\varepsilon_2^{(n)})_{[1]}$ are defined in Appendix B. Note that, by Proposition 6.26, the operator $\boldsymbol{I}_1$ coincides (up to a constant shift) with the Virasoro operator L_0 for $\widehat{\mathfrak{gl}}_k$. By using the description in Section 4.4, these operators can be written partly in terms of the Heisenberg operators $\mathfrak{p}_m^i$ of $\mathfrak{h}_{\mathbb{H}_1}$ and the $\widehat{\mathfrak{sl}}_k$ generators $\mathfrak{q}_0^i=h_i$; one has

$$\begin{aligned}\boldsymbol{I}_1&=\sum_{i=1}^{k}\beta_i\sum_{m=1}^{\infty}\mathfrak{p}_{-m}^i\,\mathfrak{p}_m^i+\frac{1}{2}\sum_{i=1}^{k-1}\big(\mathfrak{q}_0^{\eta_i}\big)^2-\frac{1}{2k}\,j\,(k-j)\ \mathrm{id}\ ,\\ \boldsymbol{I}_2&=\sum_{i=1}^{k}\varepsilon_1^{(i)}\,\Big(\frac{\beta_i}{2}\sum_{m,n=1}^{\infty}\big(\mathfrak{p}_{-m}^i\,\mathfrak{p}_{-n}^i\,\mathfrak{p}_{m+n}^i+\mathfrak{p}_{-m-n}^i\,\mathfrak{p}_n^i\,\mathfrak{p}_m^i\big)\\ &\qquad\qquad-\frac{\beta_i-1}{2}\sum_{m=1}^{\infty}(m-1)\,\mathfrak{p}_{-m}^i\,\mathfrak{p}_m^i\Big)\\ &\quad+\sum_{i=1}^{k}\varepsilon_1^{(i)}\sum_{j=1}^{k-1}\Big(\big(C^{-1}\big)^{ij}\,\beta_i-\big(C^{-1}\big)^{i-1,j}\Big)\sum_{m=1}^{\infty}\mathfrak{p}_{-m}^i\,\mathfrak{p}_m^i\,\mathfrak{q}_0^j+\boldsymbol{L}_1\ ,\end{aligned}$$

where $\boldsymbol{L}_1$ is the operator defined by $\boldsymbol{L}_1\triangleright[\vec{Y},\vec{u}\,]:=\sum_{n=1}^{k-1}\ell_{\vec{v}}^{(n)}(\varepsilon_1^{(n)},\varepsilon_2^{(n)})_{[1]}\,[\vec{Y},\vec{u}\,]$ and we set $(C^{-1})^{0j}=0=(C^{-1})^{kj}$.

Following [7], here we shall identify a quantum integrable system for each Heisenberg subalgebra $\mathfrak{h}$ of $\widehat{\mathfrak{gl}}_k$. Then each integral of motion associated to the Heisenberg algebra $\mathfrak{h}_{\mathbb{H}_1}$ is a sum of integrals of motion of k non-interacting Calogero-Sutherland models from Section 4.4 with prescribed couplings;

in particular the Hamiltonian is given by k copies of one-component bosonized Calogero-Sutherland Hamiltonians as $\sum_{i=1}^{k} \varepsilon_1^{(i)} \square^{\beta_i^{-1}}$. This infinite system of commuting operators is diagonalized in the fixed-point basis $[\vec{Y}, \vec{u}\,]$. This simultaneous eigenbasis also factorizes the primary operators from Theorem 7.6. The remaining $\vec{v}$-dependent parts of the eigenvalues are instead interpreted as particular matrix elements of our geometrically defined vertex operators $\mathrm{V}_\mu(\vec{x}, z)$ in highest weight vectors of $\widehat{\mathfrak{gl}}_k$, as we discuss in Appendix B.

Remark 7.9. By Remark 5.14, the moduli spaces $\mathcal{M}(\vec{u}, n, j)$ are Nakajima quiver varieties of type $\widehat{A}_{k-1}$. The descriptions of the integrable systems corresponding to quiver varieties are detailed in [40, 57]. △

8 $\mathcal{N} = 2$ quiver gauge theories on X_k

8.1 $\mathcal{N} = 2$ gauge theory

In this subsection we fix $j \in \{0, 1, \ldots, k-1\}$ corresponding to a fixed holonomy at infinity. The instanton partition function for the pure $\mathcal{N} = 2$ $U(1)$ gauge theory on the ALE space X_k is the generating function (cf. [14, Section 5.1])

$$\begin{aligned} \mathcal{Z}_{X_k}\big(\varepsilon_1, \varepsilon_2; \mathsf{q}, \vec{\xi}\,\big)_j &:= \sum_{\substack{\vec{v} \in \frac{1}{k}\,\mathbb{Z}^{k-1} \\ k\, v_{k-1} = j \bmod k}} \vec{\xi}^{\,\vec{v}} \sum_{n=0}^{\infty} \mathsf{q}^{n + \frac{1}{2}\, \vec{v} \cdot C \vec{v}} \int_{\mathcal{M}(\vec{u}, n, j)} \big[\mathcal{M}(\vec{u}, n, j)\big]_T \\ &= \sum_{\vec{u} \in \mathfrak{U}_j} \sum_{n=0}^{\infty} \mathsf{q}^{\frac{1}{2}\, \vec{u} \cdot C^{-1} \vec{u}}\, \vec{\xi}^{\,C^{-1} \vec{u}}\, (-\mathsf{q})^n \big\langle [\mathcal{M}(\vec{u}, n, j)]_T \,,\, [\mathcal{M}(\vec{u}, n, j)]_T \big\rangle_{\mathbb{W}_j} \,, \end{aligned}$$

where $\mathsf{q} \in \mathbb{C}^*$ with $|\mathsf{q}| < 1$, and the fugacities $\vec{\xi} := (\xi_1, \ldots, \xi_{k-1}) \in (\mathbb{C}^*)^{k-1}$ with $|\xi_i| < 1$ can be interpreted as coordinates on the maximal torus of the Lie group $SL(k, \mathbb{C})$.

In general, as described in [14, Section 5.1], the partition functions factorize into products of the corresponding instanton partition functions over the affine toric subsets $U_i \simeq \mathbb{C}^2$ of X_k and are weighted by the edge contributions $\ell_{\vec{v}}^{(n)}$ which appear in the equivariant Euler classes of the Carlsson-Okounkov bundle from Section 7.1 (see Appendix B). The edge contributions for the rank one $\mathcal{N} = 2$ gauge theory on X_k are roughly speaking the equivariant Euler classes of the vector space $H^1(\mathscr{X}_k, \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))$, which are zero by the computation of the rank of the natural bundle in [14, Appendix A], hence the edge contribution is always equal to one. We thus obtain a factorization in terms of Nekrasov partition functions for the pure $\mathcal{N} = 2$ gauge theory on $\mathbb{R}^4$ given by

$$\mathcal{Z}_{X_k}\big(\varepsilon_1, \varepsilon_2; \mathsf{q}, \vec{\xi}\,\big)_j = \sum_{\vec{u} \in \mathfrak{U}_j} \mathsf{q}^{\frac{1}{2}\, \vec{u} \cdot C^{-1} \vec{u}}\, \vec{\xi}^{\,C^{-1} \vec{u}} \prod_{i=1}^{k} \mathcal{Z}_{\mathbb{C}^2}\big(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}; \mathsf{q}\big) \,.$$

Let us denote by

$$\chi^{\widehat{\omega}_j}\big(\mathsf{q}, \vec{\zeta}\,\big) := \mathrm{Tr}_{\mathcal{V}(\widehat{\omega}_j)}\, \mathsf{q}^{L_0^{\widehat{\mathfrak{sl}}_k} - \frac{k-1}{24}\, \mathrm{id}}\, \vec{x}^{\,\vec{h}} \tag{8.1}$$

the character of the j-th dominant highest weight representation of $\widehat{\mathfrak{sl}}_k$ at level one, with weight the j-th fundamental weight $\widehat{\omega}_j$ of type $\widehat{A}_{k-1}$ for $j = 0, 1, \ldots, k-1$; here $\vec{\zeta} := \sum_{i=1}^{k-1} z_i\, H_i$ and

$x_i := \mathrm{e}^{2\pi\,\mathrm{i}\, z_i}$ for $i = 1, \dots, k-1$, while $\vec{h} := (h_1, \dots, h_{k-1})$. Setting $\xi_i = \mathrm{e}^{2\pi\,\mathrm{i}\,(2z_i - z_{i-1} - z_{i+1})}$ for $i = 1, \dots, k-1$ with $z_0 = z_k = 0$, by explicit computation of the character we get [14, Section 5.3]

$$\chi^{\widehat{\omega}_j}\big(\mathsf{q}, \vec{\zeta}\,\big) = \frac{1}{\eta(\mathsf{q})^{k-1}} \sum_{\vec{u}\in\mathfrak{U}_j} \mathsf{q}^{\frac{1}{2}\,\vec{u}\cdot C^{-1}\vec{u}}\ \vec{\xi}^{\ C^{-1}\vec{u}} ,$$

where $\mathsf{q}^{\frac{1}{24}}\,\eta(\mathsf{q})^{-1} = \prod_{n=1}^{\infty} (1-q^n)^{-1} = \mathrm{Tr}_{\mathcal{F}_{\mathbb{C}(\varepsilon_1,\varepsilon_2)}}\ \mathsf{q}^{L_0^{\mathfrak{h}}}$ is the character of the Fock space representation of the Heisenberg algebra $\mathfrak{h}$. By Equation (4.23) and the identity

$$\sum_{i=1}^{k} \frac{1}{\varepsilon_1^{(i)}\ \varepsilon_2^{(i)}} = \frac{1}{k\,\varepsilon_1\,\varepsilon_2} \tag{8.2}$$

we obtain explicitly

$$\mathcal{Z}_{X_k}\big(\varepsilon_1, \varepsilon_2; \mathsf{q}, \vec{\xi}\,\big)_j = \eta(\mathsf{q})^{k-1}\ \chi^{\widehat{\omega}_j}\big(\mathsf{q}, \vec{\zeta}\,\big)\ \exp\Big(\frac{\mathsf{q}}{k\,\varepsilon_1\,\varepsilon_2}\Big) .$$

8.1.1 Gaiotto state

Following Section 4.5.1, we define the *Gaiotto state* G_j to be the sum, in the completed total equivariant cohomology $\widehat{\mathbb{W}}_j$, of all fundamental classes

$$G_j := \sum_{\vec{u}\in\mathfrak{U}_j} \sum_{n\geq 0} \big[\mathcal{M}(\vec{u}, n, j)\big]_T .$$

We also define the *weighted Gaiotto state*

$$G_j\big(\mathsf{q}, \vec{\xi}\,\big) := \sum_{\vec{u}\in\mathfrak{U}_j} \sum_{n\geq 0} \mathsf{q}^{n+\frac{1}{2}\,\vec{u}\cdot C^{-1}\vec{u}}\ \vec{\xi}^{\ C^{-1}\vec{u}}\ \big[\mathcal{M}(\vec{u}, n, j)\big]_T$$

in the completion

$$\widehat{\mathbb{W}}_j\big(\mathsf{q}, \vec{\xi}\,\big) := \prod_{\vec{u}\in\mathfrak{U}_j} \prod_{n\geq 0} \mathsf{q}^{n+\frac{1}{2}\,\vec{u}\cdot C^{-1}\vec{u}}\ \vec{\xi}^{\ C^{-1}\vec{u}}\ \mathbb{W}_{\vec{u},n,j} .$$

If we endow $\widehat{\mathbb{W}}_j\big(\mathsf{q}, \vec{\xi}\,\big)$ with the scalar product

$$\begin{aligned} \Big\langle \sum_{\vec{u}\in\mathfrak{U}_j} \sum_{n\geq 0} \mathsf{q}^{n+\frac{1}{2}\,\vec{u}\cdot C^{-1}\vec{u}}\ \vec{\xi}^{\ C^{-1}\vec{u}}\ \eta_{\vec{u},n}\,,\ \sum_{\vec{u}\in\mathfrak{U}_j} \sum_{n\geq 0} \mathsf{q}^{n+\frac{1}{2}\,\vec{u}\cdot C^{-1}\vec{u}}\ \vec{\xi}^{\ C^{-1}\vec{u}}\ \nu_{\vec{u},n} \Big\rangle_{\widehat{\mathbb{W}}_j(\mathsf{q},\vec{\xi}\,)} \\ := \sum_{\vec{u}\in\mathfrak{U}_j} \sum_{n=0}^{\infty} \mathsf{q}^{\frac{1}{2}\,\vec{u}\cdot C^{-1}\vec{u}}\ (-\mathsf{q})^n\ \vec{\xi}^{\ C^{-1}\vec{u}}\ \big\langle \eta_{\vec{u},n}, \nu_{\vec{u},n}\big\rangle_{\mathbb{W}_{\vec{u},n,j}} , \end{aligned}$$

then it is straightforward to see that the norm of the weighted Gaiotto state is the instanton partition function for the $\mathcal{N}=2$ $U(1)$ gauge theory on X_k:

$$\mathcal{Z}_{X_k}\big(\varepsilon_1, \varepsilon_2; \mathsf{q}, \vec{\xi}\,\big)_j = \big\langle G_j\big(\mathsf{q}, \vec{\xi}\,\big)\,,\, G_j\big(\mathsf{q}, \vec{\xi}\,\big)\big\rangle_{\widehat{\mathbb{W}}_j(\mathsf{q},\vec{\xi}\,)} .$$

Proposition 8.3. *The Gaiotto state is a Whittaker vector for $\widehat{\mathfrak{gl}}_k$ of type χ, where the algebra homomorphism $\chi\colon \mathcal{U}(\mathfrak{h}^+ \oplus \mathfrak{h}^+_{\mathbb{C}(\varepsilon_1,\varepsilon_2),\mathfrak{Q}}) \to \mathbb{C}(\varepsilon_1, \varepsilon_2)$ is defined by*

$$\chi(\mathfrak{q}^i_m) = 0 \qquad \textit{for} \quad m > 0\,,\ i = 1, \dots, k-1\,,$$

$$\chi(\mathfrak{p}_m) = \delta_{m,1}\, \sqrt{\big\langle [X_k], [X_k]\big\rangle_{\mathbb{H}_1}} \qquad \textit{for} \quad m > 0\,.$$

Proof. We first note that under the isomorphism Ψ_j defined in (6.2) the fundamental class $[\mathcal{M}(\vec{u},n,j)]_T$ is sent to $[\mathrm{Hilb}^n(X_k)]_T \otimes (\gamma_{\vec{u}} + \omega_j)$. Hence under the isomorphism (6.8) the Gaiotto state becomes

$$G_j = \bigotimes_{i=1}^{k} \sum_{n\geq 0} \big[\mathrm{Hilb}^n(U_i)\big]_T \otimes \sum_{\vec{u}\in\mathfrak{U}_j} (\gamma_{\vec{u}} + \omega_j) \in \bigotimes_{i=1}^{k} \widehat{\mathbb{H}}_{U_i} \otimes \prod_{\vec{u}\in\mathfrak{U}_j} \mathbb{C}(\varepsilon_1,\varepsilon_2)(\gamma_{\vec{u}} + \omega_j) \ .$$

By Proposition 4.24, $\sum_{n\geq 0} [\mathrm{Hilb}^n(U_i)]_T$ is the Whittaker vector $G(\eta_i)$ for the Heisenberg algebra $\mathfrak{h}_i$ with $\eta_i = \big(\varepsilon_2^{(i)}\big)^{-1}$. It follows that G_j is the Whittaker vector $G_j(\vec{\eta})$ for $\widehat{\mathfrak{gl}}_k$ as in Proposition 6.29 with $\vec{\eta} = (\eta_1,\dots,\eta_k)$ of type χ where

$$\chi(\mathfrak{q}^i_m) = \delta_{m,1}\,\big(\eta_{i+1}\,\beta_{i+1}^{-1} - \eta_i\big) \qquad \text{and} \qquad \chi(\mathfrak{p}_m) = \delta_{m,1}\,\sqrt{-k\,\varepsilon_1\,\varepsilon_2}\,\sum_{i=1}^{k} \frac{\eta_i}{\varepsilon_1^{(i)}} \ .$$

By computing explicitly the quantities on the right-hand sides of these equations, one gets the assertion. □

8.2 Quiver gauge theories

As we did in Section 4.6, we will now add matter to the $\mathcal{N}=2$ gauge theory and consider $\mathcal{N}=2$ superconformal quiver gauge theories on the ALE space X_k with gauge group $U(1)^{r+1}$ for $r\geq 0$; we shall follow [16], where superconformal quiver gauge theories on the ALE space X_k are introduced.

For a quiver $\mathtt{Q} = (\mathtt{Q}_0,\mathtt{Q}_1)$ we fix vectors of integers $(\vec{u}_v,n_v,j_v)_{v\in\mathtt{Q}_0}$ representing the topological numbers of the moduli spaces $\mathcal{M}(\vec{u}_v,n_v,j_v)$ at the vertices $\mathtt{Q}_0$ with $\vec{u}_v\in\mathfrak{U}_{j_v}$, $n_v\in\mathbb{N}$, and $j_v\in\{0,1,\dots,k-1\}$. The fundamental (resp. antifundamental) hypermultiplets of masses μ_v^s, $s=1,\dots,m_v$ (resp. $\bar{\mu}_v^{\bar{s}}$, $\bar{s}=1,\dots,\bar{m}_v$) at the nodes $v\in\mathtt{Q}_0$ correspond to the T-equivariant vector bundles $\boldsymbol{V}_{\mu_v^s}^{\vec{u}_v,n_v,j_v}$ (resp. $\bar{\boldsymbol{V}}_{\bar{\mu}_v^{\bar{s}}}^{\vec{u}_v,n_v,j_v}$) on $\mathcal{M}(\vec{u}_v,n_v,j_v)$ obtained by pushforward of $\boldsymbol{E}_{\mu_v^s}^{\vec{u}_v,n_v,j_v;\vec{u}_v,n_v,j_v}$ (resp. $\boldsymbol{E}_{\bar{\mu}_v^{\bar{s}}}^{\vec{u}_v,n_v,j_v;\vec{u}_v,n_v,j_v}$) with respect to the projection of $\mathcal{M}(\vec{u}_v,n_v,j_v)\times\mathcal{M}(\vec{u}_v,n_v,j_v)$ to the second (resp. first) factor. The bifundamental hypermultiplets of masses μ_e at the edges $e\in\mathtt{Q}_1$ correspond to the vector bundles $\boldsymbol{E}_{\mu_e}^{\vec{u}_{\mathtt{s}(e)},n_{\mathtt{s}(e)},j_{\mathtt{s}(e)};\vec{u}_{\mathtt{t}(e)},n_{\mathtt{t}(e)},j_{\mathtt{t}(e)}}$ on $\mathcal{M}(\vec{u}_{\mathtt{s}(e)},n_{\mathtt{s}(e)},j_{\mathtt{s}(e)})\times\mathcal{M}(\vec{u}_{\mathtt{t}(e)},n_{\mathtt{t}(e)},j_{\mathtt{t}(e)})$; for vertex loops with $\mathtt{s}(e)=\mathtt{t}(e)$ the restriction of $\boldsymbol{E}_{\mu_e}^{\vec{u}_{\mathtt{s}(e)},n_{\mathtt{s}(e)},j_{\mathtt{s}(e)};\vec{u}_{\mathtt{s}(e)},n_{\mathtt{s}(e)},j_{\mathtt{s}(e)}}$ to the diagonal of $\mathcal{M}(\vec{u}_{\mathtt{s}(e)},n_{\mathtt{s}(e)},j_{\mathtt{s}(e)})\times\mathcal{M}(\vec{u}_{\mathtt{s}(e)},n_{\mathtt{s}(e)},j_{\mathtt{s}(e)})$ describes an adjoint hypermultiplet of mass μ_e. The total matter field content of the $\mathcal{N}=2$ quiver gauge theory on X_k associated to $\mathtt{Q}$ in the sector labelled by $(\vec{u}_v,n_v,j_v)_{v\in\mathtt{Q}_0}$ is thus described by the bundle on $\prod_{v\in\mathtt{Q}_0}\mathcal{M}(\vec{u}_v,n_v,j_v)$ given by

$$\begin{aligned}\boldsymbol{M}^{(\vec{u}_v,n_v,j_v)}_{(\mu_v^s),(\bar{\mu}_v^{\bar{s}}),(\mu_e)} := \bigoplus_{v\in\mathtt{Q}_0} p_v^*\Big(\bigoplus_{s=1}^{m_v}\boldsymbol{V}_{\mu_v^s}^{\vec{u}_v,n_v,j_v} \oplus \bigoplus_{\bar{s}=1}^{\bar{m}_v}\bar{\boldsymbol{V}}_{\bar{\mu}_v^{\bar{s}}}^{\vec{u}_v,n_v,j_v}\Big)\\ \oplus \bigoplus_{e\in\mathtt{Q}_1} p_e^*\boldsymbol{E}_{\mu_e}^{\vec{u}_{\mathtt{s}(e)},n_{\mathtt{s}(e)},j_{\mathtt{s}(e)};\vec{u}_{\mathtt{t}(e)},n_{\mathtt{t}(e)},j_{\mathtt{t}(e)}}\ ,\end{aligned}$$

where p_v is the projection of $\prod_{v\in\mathtt{Q}_0}\mathcal{M}(\vec{u}_v,n_v,j_v)$ to the v-th factor while p_e is the projection to the product $\mathcal{M}(\vec{u}_{\mathtt{s}(e)},n_{\mathtt{s}(e)},j_{\mathtt{s}(e)})\times\mathcal{M}(\vec{u}_{\mathtt{t}(e)},n_{\mathtt{t}(e)},j_{\mathtt{t}(e)})$.

The degree of the Euler class of the hypermultiplet bundle $\boldsymbol{M}^{(\vec{u}_v,n_v,j_v)}_{(\mu_v^s),(\bar{\mu}_v^{\bar{s}}),(\mu_e)}$ is given by

$$\deg\,\mathrm{eu}\,\big(\boldsymbol{M}^{(\vec{u}_v,n_v,j_v)}_{(\mu_v^s),(\bar{\mu}_v^{\bar{s}}),(\mu_e)}\big) := \sum_{v\in\mathtt{Q}_0}\dim\mathcal{M}(\vec{u}_v,n_v,j_v) - \mathrm{rk}\,\big(\boldsymbol{M}^{(\vec{u}_v,n_v,j_v)}_{(\mu_v^s),(\bar{\mu}_v^{\bar{s}}),(\mu_e)}\big)$$

$$
\begin{aligned}
&= \sum_{\upsilon\in \mathsf{Q}_0} 2\, n_\upsilon - \sum_{\upsilon\in \mathsf{Q}_0} \big(m_\upsilon + \bar m_\upsilon\big)\Big(n_\upsilon + \tfrac12\,\vec v_\upsilon\cdot C\vec v_\upsilon - \tfrac{1}{2k}\, j_\upsilon\,(k-j_\upsilon)\Big)\\
&-\sum_{e\in \mathsf{Q}_1}\Big(n_{\mathtt{t}(e)}+n_{\mathtt{s}(e)}+\tfrac12\,\vec v_{\mathtt{s}(e)}\cdot C\vec v_{\mathtt{s}(e)}+\tfrac12\,\vec v_{\mathtt{t}(e)}\cdot C\vec v_{\mathtt{t}(e)}-\vec v_{\mathtt{s}(e)}\cdot C\vec v_{\mathtt{t}(e)}-\tfrac{1}{2k}\, j_{\mathtt{t}(e)\mathtt{s}(e)}\,(k-j_{\mathtt{t}(e)\mathtt{s}(e)})\Big)\ ,
\end{aligned}
$$

where $\vec v_\upsilon := C^{-1}\vec u_\upsilon$. Using (4.26) the degree becomes

$$
\deg\, \mathrm{eu}\big(\boldsymbol{M}^{(\vec u_\upsilon,n_\upsilon,j_\upsilon)}_{(\mu^s_\upsilon),(\bar\mu^{\bar s}_\upsilon),(\mu_e)}\big) = \sum_{\upsilon\in\mathsf{Q}_0} d^{X_k}_\upsilon(\boldsymbol{\vec v}_\upsilon)\ ,
$$

where we defined

$$
\begin{aligned}
d^{X_k}_\upsilon(\boldsymbol{\vec v}_\upsilon) := \tfrac{1}{2k}\, j_\upsilon\,(k-j_\upsilon)\,\big(2-\#\{e\in\mathsf{Q}_1\,|\,\mathtt{s}(e)=\upsilon\}-\#\{e\in\mathsf{Q}_1\,|\,\mathtt{t}(e)=\upsilon\}\big)\\
+\tfrac{1}{4k}\Big(\sum_{\substack{e\in\mathsf{Q}_1\\ \mathtt{s}(e)=\upsilon}} j_{\mathtt{t}(e)\upsilon}\,(k-j_{\mathtt{t}(e)\upsilon})+\sum_{\substack{e\in\mathsf{Q}_1\\ \mathtt{t}(e)=\upsilon}} j_{\upsilon\mathtt{s}(e)}\,(k-j_{\upsilon\mathtt{s}(e)})\Big)\\
-\vec v_\upsilon\cdot C\vec v_\upsilon+\tfrac12\sum_{\substack{e\in\mathsf{Q}_1\\ \mathtt{s}(e)=\upsilon}}\vec v_\upsilon\cdot C\vec v_{\mathtt{t}(e)}+\tfrac12\sum_{\substack{e\in\mathsf{Q}_1\\ \mathtt{t}(e)=\upsilon}}\vec v_\upsilon\cdot C\vec v_{\mathtt{s}(e)}
\end{aligned}
$$

for each vertex $\upsilon\in\mathsf{Q}_0$; here $\boldsymbol{\vec v}_\upsilon := \big(\vec v_\upsilon,(\vec v_{\mathtt{t}(e)})_{e\in\mathsf{Q}_1\,:\,\mathtt{s}(e)=\upsilon},(\vec v_{\mathtt{s}(e)})_{e\in\mathsf{Q}_1\,:\,\mathtt{t}(e)=\upsilon}\big)$. By analogy with the case of gauge theories on $\mathbb{R}^4$ (see Section 4.6), we say that the $\mathcal{N}=2$ quiver gauge theory on X_k is conformal if $d^{X_k}_\upsilon(\boldsymbol{\vec v}_\upsilon)=0$ for all $\upsilon\in\mathsf{Q}_0$; this is formally the requirement of vanishing beta-function for the running of the υ-th gauge coupling constant. For any vertex $\upsilon\in\mathsf{Q}_0$, define the set of conformal fractional instanton charges by

$$
\mathfrak{U}^{\rm conf}_{j_\upsilon} := \big\{\vec u_\upsilon\in\mathfrak{U}_{j_\upsilon}\ \big|\ d^{X_k}_\upsilon(\boldsymbol{\vec v}_\upsilon)=0\big\}\ . \tag{8.4}
$$

The conformal constraint is always trivially satisfied by any $\vec u_\upsilon$ for the $\widehat{A}_0$-theory, while for the A_0-theory the set of conformal fractional instantons charges reduces to

$$
\mathfrak{U}^{\rm conf}_{j} := \big\{\vec u\in\mathfrak{U}_{j}\ \big|\ \vec u\cdot C^{-1}\vec u=\tfrac1k\, j\,(k-j)\big\}\ .
$$

Note that in this case, this is a restriction on the conformal dimension $\Delta_{\vec u}=\frac12\,\langle\omega_j,\omega_j\rangle_{\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{Q}}$ of the highest weight representation $\mathbb{W}_{\vec u,j}$ of the Virasoro algebra.

Introduce topological couplings $\mathsf{q}_\upsilon\in\mathbb{C}^*$ with $|\mathsf{q}_\upsilon|<1$ and $\vec\xi_\upsilon=\big((\xi_\upsilon)_1,\dots,(\xi_\upsilon)_{k-1}\big)\in(\mathbb{C}^*)^{k-1}$ with $|(\xi_\upsilon)_i|<1$ at each vertex $\upsilon\in\mathsf{Q}_0$. With notation as in Section 4.6, the quiver gauge theory partition function on X_k is then defined by the generating function

$$
\begin{aligned}
\mathcal{Z}^{\mathsf{Q}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathbf{q},\boldsymbol{\vec\xi}\,\big)_{\boldsymbol{j}} &:= \sum_{(\vec u_\upsilon\in\mathfrak{U}^{\rm conf}_{j_\upsilon})}\boldsymbol{\vec\xi}^{\;C^{-1}\vec{\boldsymbol{u}}}\sum_{(n_\upsilon)\in\mathbb{N}^{\mathsf{Q}_0}}\mathbf{q}^{\boldsymbol{n}+\frac12\,\vec{\boldsymbol{u}}\cdot C^{-1}\vec{\boldsymbol{u}}}\\
&\qquad\times\int_{\prod_{\upsilon\in\mathsf{Q}_0}\mathcal{M}(\vec u_\upsilon,n_\upsilon,j_\upsilon)}\mathrm{eu}_{T\times T_{\boldsymbol{\mu}}}\big(\boldsymbol{M}^{(\vec u_\upsilon,n_\upsilon,j_\upsilon)}_{(\mu^s_\upsilon),(\bar\mu^{\bar s}_\upsilon),(\mu_e)}\big)\\
&= \sum_{(\vec u_\upsilon\in\mathfrak{U}^{\rm conf}_{j_\upsilon})}\ \sum_{(n_\upsilon)\in\mathbb{N}^{\mathsf{Q}_0}}\mathbf{q}^{\boldsymbol{n}+\frac12\,\vec{\boldsymbol{u}}\cdot C^{-1}\vec{\boldsymbol{u}}}\,\boldsymbol{\vec\xi}^{\;C^{-1}\vec{\boldsymbol{u}}}
\end{aligned}
$$

$$\times \int_{\prod_{v\in\mathsf{Q}_0}\mathcal{M}(\vec{u}_v,n_v,j_v)} \prod_{v\in\mathsf{Q}_0} p_v^*\Big(\prod_{s=1}^{m_v} \mathrm{eu}_T\big(\boldsymbol{V}_{\mu_v^s}^{\vec{u}_v,n_v,j_v}\big) \prod_{\bar{s}=1}^{\bar{m}_v} \mathrm{eu}_T\big(\bar{\boldsymbol{V}}_{\bar{\mu}_v^{\bar{s}}}^{\vec{u}_v,n_v,j_v}\big)\Big)$$
$$\times \prod_{e\in\mathsf{Q}_1} p_e^*\ \mathrm{eu}_T\big(\boldsymbol{E}_{\mu_e}^{\vec{u}_{\mathsf{s}(e)},n_{\mathsf{s}(e)},j_{\mathsf{s}(e)};\vec{u}_{\mathsf{t}(e)},n_{\mathsf{t}(e)},j_{\mathsf{t}(e)}}\big)\ ,$$

where $\mathsf{q}^{\boldsymbol{n}+\frac{1}{2}\vec{\boldsymbol{u}}\cdot C^{-1}\vec{\boldsymbol{u}}} := \prod_{v\in\mathsf{Q}_0}\mathsf{q}_v^{n_v+\frac{1}{2}\vec{u}_v\cdot C^{-1}\vec{u}_v}$ and $\vec{\boldsymbol{\xi}}^{\,C^{-1}\vec{\boldsymbol{u}}} := \prod_{v\in\mathsf{Q}_0}\vec{\xi}_v^{\,C^{-1}\vec{u}_v}$. By applying the localization theorem, and using Equations (4.25) and (7.3), we obtain a factorization in terms of $\mathcal{N}=2$ quiver gauge theory partition functions on $\mathbb{R}^4$ weighted by edge contributions. For the fundamental and antifundamental matter fields, the relevant edge contributions $\ell_{\vec{v}_v}^{(n)}$ are the equivariant Euler classes of $H^1(\mathscr{X}_k,\mathcal{R}^{\vec{u}_v}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))$ for $v\in\mathsf{Q}_0$; by [14, Section 5] this vector space is zero if and only if $\vec{u}_v\in\mathfrak{U}_{j_v}^{\mathrm{conf}}$ and the corresponding edge contribution is equal to one. Thus only the arrows of the quiver yield edge contributions and the partition function is given by

$$\mathcal{Z}_{X_k}^{\mathsf{Q}}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathsf{q},\vec{\boldsymbol{\xi}}\,\big)_{\boldsymbol{j}} = \prod_{v\in\mathsf{Q}_0}\mathsf{q}_v^{\frac{1}{2}\langle\omega_{j_v},\omega_{j_v}\rangle_{\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{Q}}} \sum_{(\vec{u}_v\in\mathfrak{U}_{j_v}^{\mathrm{conf}})} \vec{\boldsymbol{\xi}}^{\,C^{-1}\vec{\boldsymbol{u}}} \prod_{n=1}^{k-1}\prod_{e\in\mathsf{Q}_1}\ell_{\vec{v}_e}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},\mu_e\big)$$
$$\times\prod_{i=1}^{k}\mathcal{Z}_{\mathbb{C}^2}^{\mathsf{Q}}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\boldsymbol{\mu}^{(i)};\mathsf{q}\big)\ ,$$

where $\vec{v}_e := \vec{v}_{\mathsf{t}(e)}-\vec{v}_{\mathsf{s}(e)}$; the shifted masses are

$$\big(\mu_v^s\big)^{(i)} := \mu_v^s-(\vec{v}_v)_i\,\varepsilon_1^{(i)}-(\vec{v}_v)_{i-1}\,\varepsilon_2^{(i)}$$

for $v\in\mathsf{Q}_0$, $s=1,\dots,m_v$ and $i=1,\dots,k$, and similarly for $\big(\bar{\mu}_v^{\bar{s}}\big)^{(i)}$, whereas

$$\mu_e^{(i)} := \mu_e-(\vec{v}_e)_i\,\varepsilon_1^{(i)}-(\vec{v}_e)_{i-1}\,\varepsilon_2^{(i)}$$

for $e\in\mathsf{Q}_1$ and $i=1,\dots,k$.

In the remainder of this section we consider in detail each of the admissible quivers in turn.

8.3 $\widehat{A}_r$ theories

With the conventions of Section 4.7, the instanton partition function for the $\mathcal{N}=2$ $U(1)^{r+1}$ quiver gauge theory of type $\widehat{A}_r$ on the ALE space X_k reads as

$$\mathcal{Z}_{X_k}^{\widehat{A}_r}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathsf{q},\vec{\boldsymbol{\xi}}\,\big)_{\boldsymbol{j}} = \prod_{v=0}^{r}\mathsf{q}_v^{\frac{1}{2}\langle\omega_{j_v},\omega_{j_v}\rangle_{\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{Q}}} \sum_{(\vec{u}_v\in\mathfrak{U}_{j_v}^{\mathrm{conf}})} \vec{\boldsymbol{\xi}}^{\,C^{-1}\vec{\boldsymbol{u}}} \prod_{n=1}^{k-1}\prod_{v=0}^{r}\ell_{\vec{v}_{v,v+1}}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},\mu_v\big)$$
$$\times\prod_{i=1}^{k}\mathcal{Z}_{\mathbb{C}^2}^{\widehat{A}_r}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\boldsymbol{\mu}^{(i)};\mathsf{q}\big)\ , \quad (8.5)$$

where $\vec{\boldsymbol{\xi}}^{\,C^{-1}\vec{\boldsymbol{u}}} := \prod_{v=0}^{r}\vec{\xi}_v^{\,C^{-1}\vec{u}_v}$ with $\vec{u}_{r+1} := \vec{u}_0$, while $\mu_v^{(i)} := \mu_v-(\vec{v}_{v,v+1})_i\,\varepsilon_1^{(i)}-(\vec{v}_{v,v+1})_{i-1}\,\varepsilon_2^{(i)}$ with $\vec{v}_{v,v+1} := \vec{v}_{v+1}-\vec{v}_v$.

8.3.1 Conformal blocks

We will relate the partition function (8.5) to the trace of vertex operators $\mathrm{V}_{\mu_\upsilon}^{j_\upsilon,j_{\upsilon+1}}(\vec{x}_\upsilon,z_\upsilon)$ from Section 7, analogously to what we did in Section 4.7, and hence interpret it as a torus $(r+1)$-point conformal block. For this, we fix vertices $\upsilon,\upsilon'\in\{0,1,\dots,r\}$ and introduce the *conformal restriction operators* $\delta_{\upsilon,\upsilon'}^{\rm conf}:\mathbb{W}\to\mathbb{W}$ which are defined by their matrix elements in the fixed point basis of the vector space $\mathbb{W}$ by

$$\big\langle\delta_{\upsilon,\upsilon'}^{\rm conf}\triangleright[\vec{Y},\vec{u}\,]\,,\,[\vec{Y}',\vec{u}\,']\big\rangle_{\mathbb{W}}:=\begin{cases}1 & \text{if}\quad \vec{u}\in\mathfrak{U}_{j_\upsilon}^{\rm conf}\,,\ \vec{u}\,'\in\mathfrak{U}_{j_{\upsilon'}}^{\rm conf}\,,\\ 0 & \text{otherwise}\,.\end{cases}\tag{8.6}$$

Suitable insertions of this operator restrict the first Chern classes $\vec{u}_\upsilon\in\mathfrak{U}_{j_\upsilon}$ in the way required by the superconformal constraints of the quiver gauge theory and the constrained conformal dimensions of the associated Virasoro algebras at the nodes of the $\widehat{A}_r$-type quivers. Using Proposition 6.26 and Proposition 6.28, by performing analogous manipulations to those used in the proof of Proposition 4.28 we arrive at the following result.

Proposition 8.7. *The partition function of the $\widehat{A}_r$-theory on X_k is given by*

$$\mathcal{Z}_{X_k}^{\widehat{A}_r}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathsf{q},\vec{\boldsymbol{\xi}}\,\big)_{\boldsymbol{j}}=\mathrm{Tr}_{\mathbb{W}_{j_0}}\ \mathsf{q}^{L_0}\ \vec{\xi}^{\,C^{-1}\vec{h}}\ \prod_{\upsilon=0}^{r}\mathrm{V}_{\mu_\upsilon}^{j_\upsilon,j_{\upsilon+1}}(\vec{x}_\upsilon,z_\upsilon)\ \delta_{\upsilon,\upsilon+1}^{\rm conf}$$

independently of $z_0\in\mathbb{C}^*$ and $\vec{x}_0\in(\mathbb{C}^*)^{k-1}$, where $\mathsf{q}:=\mathsf{q}_0\,\mathsf{q}_1\cdots\mathsf{q}_r$, $(\,\vec{\xi}\,)_i:=(\vec{\xi}_0)_i\,(\vec{\xi}_1)_i\cdots(\vec{\xi}_r)_i$, $z_\upsilon:=z_0\,\mathsf{q}_1\cdots\mathsf{q}_\upsilon$, and $(\vec{x}_\upsilon)_i:=(\vec{x}_0)_i\,(\vec{\xi}_1)_i\cdots(\vec{\xi}_\upsilon)_i$ for $\upsilon=1,\dots,r$ and $i=1,\dots,k-1$.

By combining Theorem 7.6 and Proposition 8.7, it follows that the quiver gauge theory partition function completely factorizes under the isomorphism of Proposition 6.24 into partition functions associated to the affine algebras $\mathfrak{h}$ and $\widehat{\mathfrak{sl}}_k$.

Corollary 8.8. *Let $\mathrm{V}_\mu(\vec{v}_{21},\vec{x},z)$ be the vertex operator in $\mathrm{Hom}(\mathbb{W}_{j_1},\mathbb{W}_{j_2})[[z^{\pm1},x_1^{\pm1},\dots,x_{k-1}^{\pm1}]]$ given by*

$$\mathrm{V}_\mu(\vec{v}_{21},\vec{x},z):=z^{\Delta_{\vec{u}_2}-\Delta_{\vec{u}_1}}\,\bar{\mathrm{V}}_\mu(\vec{v}_{21},\vec{x},z)\ \exp\big(\log z\ \mathfrak{c}-\gamma_{21}\big)\ \exp\big(\gamma_{21}\big)\,.\tag{8.9}$$

Then

$$\begin{aligned}\mathcal{Z}_{X_k}^{\widehat{A}_r}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathsf{q},\vec{\boldsymbol{\xi}}\,\big)_{\boldsymbol{j}}&=\mathcal{Z}_{\mathbb{C}^2}^{\widehat{A}_r}(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathsf{q})^{\frac1k}\ \mathsf{q}^{\frac{1}{24}\,(1-\frac1k)}\ \eta(\mathsf{q})^{\frac1k-1}\\&\quad\times\ \mathrm{Tr}_{\mathcal{V}(\widehat{\omega}_{j_0})}\ \mathsf{q}^{L_0^{\widehat{\mathfrak{sl}}_k}}\ \vec{\xi}^{\,C^{-1}\vec{h}}\ \prod_{\upsilon=0}^{r}\ \sum_{(\vec{u}_\upsilon\in\mathfrak{U}_{j_\upsilon}^{\rm conf})}\ \mathrm{V}_{\mu_\upsilon}(\vec{v}_{\upsilon,\upsilon+1},\vec{x}_\upsilon,z_\upsilon)\,\big|_{\mathbb{W}_{\vec{u}_\upsilon,j_\upsilon}}\,.\end{aligned}$$

Proof. By Theorem 7.6 and Proposition 6.24 we get

$$\begin{aligned}\mathrm{Tr}_{\mathbb{W}_{j_0}}\ \mathsf{q}^{L_0}\ \vec{\xi}^{\,C^{-1}\vec{h}}\ \prod_{\upsilon=0}^{r}\mathrm{V}_{\mu_\upsilon}^{j_\upsilon,j_{\upsilon+1}}(\vec{x}_\upsilon,z_\upsilon)\ \delta_{\upsilon,\upsilon+1}^{\rm conf}&=\mathrm{Tr}_{\mathcal{F}_{\mathbb{C}(\varepsilon_1,\varepsilon_2)}}\ \mathsf{q}^{L_0^{\mathfrak{h}}}\ \prod_{\upsilon=0}^{r}\mathrm{V}_{-\frac{\mu_\upsilon}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}},\frac{\mu_\upsilon+\varepsilon_1+\varepsilon_2}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}}}(z_\upsilon)\\&\quad\times\ \mathrm{Tr}_{\mathcal{V}(\widehat{\omega}_{j_0})}\ \mathsf{q}^{L_0^{\widehat{\mathfrak{sl}}_k}}\ \vec{\xi}^{\,C^{-1}\vec{h}}\ \prod_{\upsilon=0}^{r}\ \sum_{\vec{u}_1\in\mathfrak{U}_{j_\upsilon},\vec{u}_2\in\mathfrak{U}_{j_{\upsilon+1}}}\mathrm{V}_\mu(\vec{v}_{21},z_\upsilon)\,\big|_{\mathbb{W}_{\vec{u}_1,j_\upsilon}}\ \delta_{\upsilon,\upsilon+1}^{\rm conf}\,.\end{aligned}$$

Then by using the same arguments as in the proof of [18, Corollary 1] one gets

$$\mathrm{Tr}_{\mathcal{F}_{\mathbb{C}(\varepsilon_1,\varepsilon_2)}}\ \mathsf{q}^{L_0^{\mathfrak{h}}}\ \prod_{v=0}^{r} \mathrm{V}_{-\frac{\mu_v}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}},\frac{\mu_v+\varepsilon_1+\varepsilon_2}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}}}(z_v) = \prod_{v=0}^{r} \big(\mathsf{q}_v^{-\frac{1}{24}}\,\eta(\mathsf{q}_v)\big)^{-\frac{\mu_v\,(\mu_v+\varepsilon_1+\varepsilon_2)}{k\,\varepsilon_1\,\varepsilon_2}}\ \mathsf{q}^{\frac{1}{24}}\,\eta(\mathsf{q})^{-1}\ .$$

The result now follows from Proposition 4.30. □

8.3.2 $\widehat{A}_0$ theory

For the $\mathcal{N}=2^*$ gauge theory on X_k, similar arguments to those of Section 8.1 show that the edge contributions are also equal to one in this case. In this instance the gauge theory is automatically conformal without further restriction of the first Chern classes $\vec{u}\in\mathfrak{U}_j$. Then the instanton partition function for $U(1)$ gauge theory on the ALE space X_k with a single adjoint hypermultiplet of mass μ can be written in a factorized form in terms of the Nekrasov partition function for $\mathcal{N}=2^*$ gauge theory on $\mathbb{R}^4$ given by

$$\mathcal{Z}_{X_k}^{\widehat{A}_0}\big(\varepsilon_1,\varepsilon_2,\mu;\mathsf{q},\vec{\xi}\,\big)_j = \eta(\mathsf{q})^{k-1}\ \chi^{\widehat{\omega}_j}\big(\mathsf{q},\vec{\zeta}\,\big)\ \prod_{i=1}^{k} \mathcal{Z}_{\mathbb{C}^2}^{\widehat{A}_0}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\mu;\mathsf{q}\big)\ .$$

In this case Proposition 8.7 may be stated in a factorized form under the decomposition of Theorem 7.6 in terms of characters of $\mathfrak{h}\subset\widehat{\mathfrak{gl}}_k$ and $\widehat{\mathfrak{gl}}_k$ as

$$\mathcal{Z}_{X_k}^{\widehat{A}_0}\big(\varepsilon_1,\varepsilon_2,\mu;\mathsf{q},\vec{\xi}\,\big)_j = \eta(\mathsf{q})^{k-1}\ \chi^{\widehat{\omega}_j}\big(\mathsf{q},\vec{\zeta}\,\big)\ \mathrm{Tr}_{\mathbb{H}}\ \mathsf{q}^{L_0^{\mathfrak{h}}}\ \mathrm{V}_{-\frac{\mu}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}},\frac{\mu+\varepsilon_1+\varepsilon_2}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}}}(1)\ .$$

By using the identities (4.31) and (8.2), we obtain explicitly

$$\mathcal{Z}_{X_k}^{\widehat{A}_0}\big(\varepsilon_1,\varepsilon_2,\mu;\mathsf{q},\vec{\xi}\,\big)_j = \mathsf{q}^{\frac{k}{24}}\ \eta(\mathsf{q})^{-1}\ \chi^{\widehat{\omega}_j}\big(\mathsf{q},\vec{\zeta}\,\big)\ \big(\mathsf{q}^{-\frac{1}{24}}\,\eta(\mathsf{q})\big)^{-\frac{\mu\,(\mu+\varepsilon_1+\varepsilon_2)}{k\,\varepsilon_1\,\varepsilon_2}}\ .$$

Remark 8.10. Note that $\mathbb{H}$ is not the Fock space of $\mathfrak{h}$, as we have

$$\mathrm{Tr}_{\mathbb{H}}\ \mathsf{q}^{L_0^{\mathfrak{h}}}\ \mathrm{V}_{-\frac{\mu}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}},\frac{\mu+\varepsilon_1+\varepsilon_2}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}}}(1) = \big(\mathsf{q}^{\frac{1}{24}}\,\eta(\mathsf{q})^{-1}\big)^{k-1}\ \mathrm{Tr}_{\mathcal{F}_{\mathbb{C}(\varepsilon_1,\varepsilon_2)}}\ \mathsf{q}^{L_0^{\mathfrak{h}}}\ \mathrm{V}_{-\frac{\mu}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}},\frac{\mu+\varepsilon_1+\varepsilon_2}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}}}(1)\ .$$

△

8.4 A_r theories

With the conventions of Section 4.8, the instanton partition function for the $\mathcal{N}=2$ $U(1)^{r+1}$ quiver gauge theory of type A_r on the ALE space X_k reads as

$$\mathcal{Z}_{X_k}^{A_r}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathbf{q},\vec{\boldsymbol{\xi}}\,\big)_{\boldsymbol{j}} = \prod_{v=0}^{r} \mathsf{q}_v^{\frac{1}{2}\,\langle\omega_{j_v},\omega_{j_v}\rangle_{\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{Q}}} \sum_{(\vec{u}_v\in\mathfrak{U}_{j_v}^{\mathrm{conf}})} \vec{\boldsymbol{\xi}}^{\,C^{-1}\vec{\boldsymbol{u}}} \prod_{n=1}^{k-1}\prod_{v=0}^{r-1} \ell_{\vec{v}_{v,v+1}}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},\mu_{v+1}\big)$$
$$\times \prod_{i=1}^{k} \mathcal{Z}_{\mathbb{C}^2}^{A_r}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\boldsymbol{\mu}^{(i)};\mathbf{q}\big)\ . \qquad (8.11)$$

8.4.1 Conformal blocks

By performing analogous manipulations to those used in the proof of Proposition 4.34, we can express the partition function (8.11) as a particular matrix element of vertex operators and hence interpret it as an $(r+4)$-point conformal block on the sphere. For this, let

$$|0\rangle_{\rm conf} := \prod_{\upsilon=0}^{r} \delta_{0,\upsilon}^{\rm conf} \triangleright [\emptyset, \vec{0}\,] .$$

Proposition 8.12. *The partition function of the A_r-theory on X_k is given by*

$$\begin{aligned}\mathcal{Z}_{X_k}^{A_r}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathsf{q},\vec{\boldsymbol{\xi}}\,\big)_{\boldsymbol{j}} \\ = \Big\langle |0\rangle_{\rm conf}\,,\, V_{\mu_0}(\vec{x}_0,z_0)\Big(\prod_{\upsilon=1}^{r} \mathrm{V}_{\mu_\upsilon}^{j_{\upsilon-1},j_\upsilon}(\vec{x}_\upsilon,z_\upsilon)\,\delta_{\upsilon-1,\upsilon}^{\rm conf}\Big)\,\mathrm{V}_{\mu_{r+1}}(\vec{x}_{r+1},z_{r+1})|0\rangle_{\rm conf}\Big\rangle_{\mathbb{W}}\end{aligned}$$

independently of $z_0\in\mathbb{C}^*$ and $\vec{x}_0\in(\mathbb{C}^*)^{k-1}$, where $z_\upsilon := z_0\,\mathsf{q}_0\,\mathsf{q}_1\cdots\mathsf{q}_\upsilon$ and $(\vec{x}_\upsilon)_i := (\vec{x}_0)_i\,(\vec{\xi}_0)_i\,(\vec{\xi}_1)_i \cdots(\vec{\xi}_{\upsilon-1})_i$ for $\upsilon=1,\dots,r+1$ and $i=1,\dots,k-1$.

Again, combining Theorem 7.6 and Proposition 8.12 yields a completely factorized form for the quiver gauge theory partition function under the isomorphism of Proposition 6.24. In the following we denote $\mathcal{V} := \bigoplus_{j=0}^{k-1}\mathcal{V}(\,\widehat{\omega}_j\,)$.

Corollary 8.13. *Let $\mathrm{V}_\mu(\vec{v}_{21},\vec{x},z)$ be the vertex operator in $\mathrm{Hom}(\mathbb{W}_{j_1},\mathbb{W}_{j_2})[[z^{\pm1},x_1^{\pm1},\dots,x_{k-1}^{\pm1}]]$ given by* (8.9)*. Then*

$$\begin{aligned}&\mathcal{Z}_{X_k}^{A_r}\big(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathsf{q},\vec{\boldsymbol{\xi}}\,\big)_{\boldsymbol{j}} = \mathcal{Z}_{\mathbb{C}^2}^{A_r}(\varepsilon_1,\varepsilon_2,\boldsymbol{\mu};\mathsf{q})^{\frac1k}\\ &\times\Big\langle|0\rangle_{\rm conf}\,,\Big(\sum_{j_0,j_0'=0}^{k-1}\ \sum_{\vec{u}_0\in\mathfrak{U}_{j_0},\vec{u}_0'\in\mathfrak{U}_{j_0'}}\mathrm{V}_{\mu_0}(\vec{v}_{0',0},\vec{x}_0,z_0)\,\big|_{\mathbb{W}_{\vec{u}_0,j_0}}\Big)\\ &\qquad\times\prod_{\upsilon=1}^{r}\ \sum_{(\vec{u}_\upsilon\in\mathfrak{U}_{j_\upsilon}^{\rm conf})}\mathrm{V}_{\mu_\upsilon}(\vec{v}_{\upsilon-1,\upsilon},\vec{x}_\upsilon,z_\upsilon)\,\big|_{\mathbb{W}_{\vec{u}_\upsilon,j_\upsilon}}\\ &\times\Big(\sum_{j_{r+1},j_{r+1}'=0}^{k-1}\ \sum_{\vec{u}_{r+1}\in\mathfrak{U}_{j_{r+1}},\vec{u}_{r+1}'\in\mathfrak{U}_{j_{r+1}'}}\mathrm{V}_{\mu_{r+1}}(\vec{v}_{r+1',r+1},\vec{x}_{r+1},z_{r+1})\,\big|_{\mathbb{W}_{\vec{u}_{r+1},j_{r+1}}}\Big)|0\rangle_{\rm conf}\Big\rangle_{\mathcal{V}}\,.\end{aligned}$$

Proof. The proof follows that of Corollary 8.8, and by repeating the proof of Proposition 4.36 to compute

$$\Big\langle|0\rangle\,,\prod_{\upsilon=0}^{r+1}\mathrm{V}_{-\frac{\mu_\upsilon}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}},\frac{\mu_\upsilon+\varepsilon_1+\varepsilon_2}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}}}(z_\upsilon)|0\rangle\Big\rangle_{\mathcal{F}_{\mathbb{C}(\varepsilon_1,\varepsilon_2)}} = \prod_{0\le\upsilon<\upsilon'\le r+1}\big(1-\mathsf{q}_{\upsilon+1}\cdots\mathsf{q}_{\upsilon'}\big)^{-\frac{\mu_{\upsilon'}\,(\mu_\upsilon+\varepsilon_1+\varepsilon_2)}{k\,\varepsilon_1\,\varepsilon_2}}\,.$$

$\square$

8.4.2 A_0 theory

For the $\mathcal{N}=2$ superconformal gauge theory on X_k with two fundamental hypermultiplets of masses μ_0, μ_1, the set $\mathfrak{U}_j^{\rm conf}$ coincides with the rank one limit of the more general conformal charge sets obtained in [14, Section 5.4]. Analogously to Equation (8.1), let us define the restricted $\widehat{\mathfrak{sl}}_k$ characters

$$\begin{aligned}\chi_{\rm conf}^{\widehat{\omega}_j}\big(\mathsf{q},\vec{\zeta}\,\big) &:= \operatorname{Tr}_{\mathcal{V}(\widehat{\omega}_j)}\ \mathsf{q}^{L_0^{\widehat{\mathfrak{sl}}_k}-\frac{k-1}{24}\ \mathrm{id}}\ \vec{x}^{\,\vec{h}}\ \delta_{j,j}^{\rm conf} \\ &= \frac{1}{\eta(\mathsf{q})^{k-1}}\ \sum_{\vec{u}\in\mathfrak{U}_j^{\rm conf}} \mathsf{q}^{\frac{1}{2}\,\vec{u}\cdot C^{-1}\vec{u}}\ \vec{\xi}^{\,C^{-1}\vec{u}} = \frac{\mathsf{q}^{\frac{1}{2}\,\langle\omega_j,\omega_j\rangle_{\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{Q}}}}{\eta(\mathsf{q})^{k-1}}\ \sum_{\vec{u}\in\mathfrak{U}_j^{\rm conf}} \vec{\xi}^{\,C^{-1}\vec{u}}\ .\end{aligned}$$

Then the instanton partition function is given by the factorization

$$\mathcal{Z}_{X_k}^{A_0}\big(\varepsilon_1,\varepsilon_2,\mu_0,\mu_1;\mathsf{q},\vec{\xi}\,\big)_j = \eta(\mathsf{q})^{k-1}\ \chi_{\rm conf}^{\widehat{\omega}_j}\big(\mathsf{q},\vec{\zeta}\,\big)\ \prod_{i=1}^{k} \mathcal{Z}_{\mathbb{C}^2}^{A_0}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\mu_0,\mu_1;\mathsf{q}\big)\ .$$

In this instance Proposition 8.12 factorizes under the decomposition of Theorem 7.6 as

$$\begin{aligned}&\mathcal{Z}_{X_k}^{A_0}\big(\varepsilon_1,\varepsilon_2,\mu_1,\mu_2;\mathsf{q},\vec{\xi}\,\big)_j \\ &\quad = \eta(\mathsf{q})^{k-1}\ \chi_{\rm conf}^{\widehat{\omega}_j}\big(\mathsf{q},\vec{\zeta}\,\big)\ \Big\langle |0\rangle\,,\ \mathrm{V}_{-\frac{\mu_0}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}},\frac{\mu_0+\varepsilon_1+\varepsilon_2}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}}}(1)\ \mathrm{V}_{-\frac{\mu_1}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}},\frac{\mu_1+\varepsilon_1+\varepsilon_2}{\sqrt{-k\,\varepsilon_1\,\varepsilon_2}}}(\mathsf{q})|0\rangle\Big\rangle_{\mathbb{H}}\ .\end{aligned}$$

By Equation (4.37) we then obtain explicitly

$$\mathcal{Z}_{X_k}^{A_0}\big(\varepsilon_1,\varepsilon_2,\mu_0,\mu_1;\mathsf{q},\vec{\xi}\,\big)_j = \eta(\mathsf{q})^{k-1}\ \chi_{\rm conf}^{\widehat{\omega}_j}\big(\mathsf{q},\vec{\zeta}\,\big)\ (1-\mathsf{q})^{-\frac{\mu_1\,(\mu_0+\varepsilon_1+\varepsilon_2)}{k\,\varepsilon_1\,\varepsilon_2}}\ .$$

A Virasoro primary fields

In this appendix we prove Theorem 7.6. We need to show that the vertex operator $\bar{\mathrm{V}}_\mu(\vec{v}_{21},\vec{x},z)$ is a primary field of the Virasoro algebra generated by $L_n^{\widehat{\mathfrak{sl}}_k}$ and $\mathfrak{c}$. We begin with the following result establishing the commutation relations between the Virasoro operators $L_n^{\widehat{\mathfrak{sl}}_k}$ introduced in Section 3.3.1 and the normal-ordered bosonic exponentials $\mathrm{V}_{1,-1}^{\gamma_{21}}(z)$ associated with the Heisenberg algebra $\mathfrak{h}_{\mathfrak{Q}}$.

Lemma A.1. *For $n\neq 0$ we have*

$$\begin{aligned}\big[L_n^{\widehat{\mathfrak{sl}}_k}\,,\,\mathrm{V}_{1,-1}^{\gamma_{21}}(z)\big] &= z^n\,\Big(z\,\partial_z+\frac{1}{2}\,\vec{v}_{21}\cdot C\vec{v}_{21}\,(n+1)+\sum_{i=1}^{k-1}\,\big(\vec{v}_{21}\big)_i\,\big(\mathsf{q}_0^i-z^{-n}\,\mathsf{q}_n^i\big)\Big)\ \mathrm{V}_{1,-1}^{\gamma_{21}}(z)\ , \\ \big[L_0^{\widehat{\mathfrak{sl}}_k}\,,\,\mathrm{V}_{1,-1}^{\gamma_{21}}(z)\big] &= z\,\partial_z\ \mathrm{V}_{1,-1}^{\gamma_{21}}(z)\ .\end{aligned}$$

Proof. Let $\{\eta_i\}_{i=1}^{k-1}$ be an orthonormal basis of the vector space $\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{R}$. By the commutation relations (3.2), we easily get

$$\big[L_n^{\widehat{\mathfrak{sl}}_k},\mathsf{q}_m^{\eta_j}\big] = -m\,\mathsf{q}_{n+m}^{\eta_j} \tag{A.2}$$

for $n, m \in \mathbb{Z}$ and $j = 1, \dots, k-1$. Fix a vector $\vec{v} \in \mathbb{R}^{k-1}$ and for $j = 1, \dots, k-1$ define

$$A^j_{\vec{v}}(z)_- := v_j\, \varphi^{\eta_j}_-(z) = v_j \sum_{m=1}^{\infty} \frac{z^m}{m}\, \mathfrak{q}^{\eta_j}_{-m} \qquad \text{and} \qquad A^j_{\vec{v}}(z)_+ := -v_j\, \varphi^{\eta_j}_+(z) = -v_j \sum_{m=1}^{\infty} \frac{z^{-m}}{m}\, \mathfrak{q}^{\eta_j}_{m} \,.$$

Using the commutation relations (A.2), we get

$$\big[L^{\widehat{\mathfrak{sl}}_k}_n, A^j_{\vec{v}}(z)_-\big] = v_j \sum_{m=1}^{\infty} z^m\, \mathfrak{q}^{\eta_j}_{n-m} =: v_j\, \varphi^{\eta_j}_{-,n}(z)\,,$$

$$\big[L^{\widehat{\mathfrak{sl}}_k}_n, A^j_{\vec{v}}(z)_+\big] = v_j \sum_{m=1}^{\infty} z^{-m}\, \mathfrak{q}^{\eta_j}_{n+m} =: v_j\, \varphi^{\eta_j}_{+,n}(z)\,.$$

For $n \le 0$ the operator $\varphi^{\eta_j}_{-,n}(z)$ is a series in the Heisenberg operators $\mathfrak{q}^{\eta_j}_l$ with $l < 0$, thus it commutes with $A^j_{\vec{v}}(z)_-$. For $n > 0$, using again the relations (3.2) we get

$$\big[A^j_{\vec{v}}(z)_-, \varphi^{\eta_j}_{-,n}(z)\big] = v_j \sum_{m,l=1}^{n-1} \frac{z^{m+l}}{m}\, \big[\mathfrak{q}^{\eta_j}_{-m}, \mathfrak{q}^{\eta_j}_{n-l}\big] = -(n-1)\, v_j\, z^n\, \mathfrak{c}\,. \tag{A.3}$$

Analogously, for $n \ge 0$ the operators $A^j_{\vec{v}}(z)_+$ and $\varphi^{\eta_j}_{+,n}(z)$ commute, while for $n < 0$ we have

$$\big[A^j_{\vec{v}}(z)_+, \varphi^{\eta_j}_{+,n}(z)\big] = (n+1)\, v_j\, z^n\, \mathfrak{c}\,.$$

Now we compute the commutator

$$\begin{aligned}\big[L^{\widehat{\mathfrak{sl}}_k}_n, \exp\big(A^j_{\vec{v}}(z)_-\big)\big] &= \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{i=0}^{l-1} A^j_{\vec{v}}(z)^i_-\, \big[L^{\widehat{\mathfrak{sl}}_k}_n, A^j_{\vec{v}}(z)_-\big]\, A^j_{\vec{v}}(z)^{l-i-1}_- \\ &= v_j \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{i=0}^{l-1} A^j_{\vec{v}}(z)^i_-\, \varphi^{\eta_j}_{-,n}(z)\, A^j_{\vec{v}}(z)^{l-i-1}_-\,.\end{aligned}$$

For $n \le 0$ we get simply $\big[L^{\widehat{\mathfrak{sl}}_k}_n, \exp\big(A^j_{\vec{v}}(z)_-\big)\big] = v_j\, \varphi^{\eta_j}_{-,n}(z)\, \exp\big(A^j_{\vec{v}}(z)_-\big)$, while for $n > 0$ we can apply (A.3) iteratively to obtain

$$\big[L^{\widehat{\mathfrak{sl}}_k}_n, \exp\big(A^j_{\vec{v}}(z)_-\big)\big] = \big(v_j\, \varphi^{\eta_j}_{-,n}(z) - \tfrac{1}{2}\, v_j^2\, (n-1)\, z^n\,\big)\, \exp\big(A^j_{\vec{v}}(z)_-\big)\,.$$

Noting that

$$z^{n+1}\, \partial_z \exp\big(A^j_{\vec{v}}(z)_-\big) = \begin{cases} v_j\, \Big(\varphi^{\eta_j}_{-,n}(z) - \sum\limits_{t=1}^{n} z^t\, \mathfrak{q}^{\eta_j}_{n-t}\Big)\, \exp\big(A^j_{\vec{v}}(z)_-\big) & n > 0 \\ v_j\, \Big(\varphi^{\eta_j}_{-,n}(z) - \sum\limits_{t=0}^{-n-1} z^{-t}\, \mathfrak{q}^{\eta_j}_{n+t}\Big)\, \exp\big(A^j_{\vec{v}}(z)_-\big) & n \le 0 \end{cases}\,,$$

and substituting in the previous expressions we finally get

$$\big[L^{\widehat{\mathfrak{sl}}_k}_n, \exp\big(A^j_{\vec{v}}(z)_-\big)\big] = \begin{cases} z^n\, \Big(z\, \partial_z + v_j \sum\limits_{s=0}^{n-1} z^{-s}\, \mathfrak{q}^{\eta_j}_s - \tfrac{1}{2}\, v_j^2\, (n-1)\Big)\, \exp\big(A^j_{\vec{v}}(z)_-\big) & n > 0 \\ z^n\, \Big(z\, \partial_z - v_j \sum\limits_{s=1}^{-n} z^s\, \mathfrak{q}^{\eta_j}_{-s}\Big)\, \exp\big(A^j_{\vec{v}}(z)_-\big) & n \le 0 \end{cases}\,.$$

Repeating these computations for the operator $A^j_{\vec v}(z)_+$, we get analogous relations

$$\big[L_n^{\widehat{\mathfrak{sl}}_k}, \exp\big(A^j_{\vec v}(z)_+\big)\big] = \begin{cases} z^n \Big(z\,\partial_z - v_j \sum\limits_{s=1}^{n} z^{-s}\,\mathfrak{q}_s^{\eta_j}\Big) \exp\big(A^j_{\vec v}(z)_+\big) & n \geq 0 \\ z^n \Big(z\,\partial_z + v_j \sum\limits_{s=0}^{-n-1} z^{s}\,\mathfrak{q}_{-s}^{\eta_j} + \frac{1}{2}\,v_j^2\,(n+1)\Big) \exp\big(A^j_{\vec v}(z)_+\big) & n < 0 \end{cases} .$$

Now set

$$A_{\vec v}(z)_- := \sum_{j=1}^{k-1} A^j_{\vec v}(z)_- \qquad \text{and} \qquad A_{\vec v}(z)_+ := \sum_{j=1}^{k-1} A^j_{\vec v}(z)_+ \ .$$

We are ready to compute

$$\begin{aligned} &\big[L_n^{\widehat{\mathfrak{sl}}_k}, \exp\big(A_{\vec v}(z)_-\big)\, \exp\big(A_{\vec v}(z)_+\big)\big] \\ &\quad = \sum_{j=1}^{k-1} \Big(\exp\big(A^1_{\vec v}(z)_-\big) \cdots \big[L_n^{\widehat{\mathfrak{sl}}_k}, \exp(A^j_{\vec v}(z)_-)\big] \cdots \exp\big(A^{k-1}_{\vec v}(z)_-\big)\, \exp\big(A_{\vec v}(z)_+\big) \\ &\qquad\qquad + \exp\big(A_{\vec v}(z)_-\big)\, \exp\big(A^1_{\vec v}(z)_+\big) \cdots \big[L_n^{\widehat{\mathfrak{sl}}_k}, \exp\big(A^j_{\vec v}(z)_+\big)\big] \cdots \exp\big(A^{k-1}_{\vec v}(z)_+\big)\Big) \ . \end{aligned}$$

Fix $n < 0$. Using the commutation relations computed before and noting that the Heisenberg operator $\mathfrak{q}_m^{\eta_j}$ commutes with the vertex operators $\exp\big(A^l_{\vec v}(z)_-\big)$ and $\exp\big(A^l_{\vec v}(z)_+\big)$ for $l \neq j$ and any $m \in \mathbb{Z} \setminus \{0\}$, we obtain

$$\begin{aligned} &\big[L_n^{\widehat{\mathfrak{sl}}_k}, \exp\big(A_{\vec v}(z)_-\big)\, \exp\big(A_{\vec v}(z)_+\big)\big] \\ &\quad = z^n \Big(z\,\partial_z + \frac{1}{2}\,(n+1) \sum_{j=1}^{k-1} v_j^2 - \sum_{j=1}^{k-1} v_j \sum_{t=1}^{-n} z^t\,\mathfrak{q}_{-t}^{\eta_j}\Big)\Big(\exp\big(A_{\vec v}(z)_-\big)\, \exp\big(A_{\vec v}(z)_+\big)\Big) \\ &\qquad\qquad + \sum_{j=1}^{k-1} v_j\, \exp\big(A_{\vec v}(z)_-\big)\, \Big(\sum_{t=0}^{-n-1} z^t\,\mathfrak{q}_{-t}^{\eta_j}\Big)\, \exp\big(A_{\vec v}(z)_+\big) \ . \end{aligned}$$

Since the Heisenberg operators $\mathfrak{q}_{-t}^{\eta_j}$ for $t \geq 0$ also commute with $\exp\big(A^j_{\vec v}(z)_-\big)$, we thus find

$$\begin{aligned} &\big[L_n^{\widehat{\mathfrak{sl}}_k}, \exp\big(A_{\vec v}(z)_-\big)\, \exp\big(A_{\vec v}(z)_+\big)\big] \\ &\quad = z^n \Big(z\,\partial_z + \frac{1}{2}\,(n+1) \sum_{j=1}^{k-1} v_j^2 - \sum_{j=1}^{k-1} v_j\,\big(\mathfrak{q}_0^{\eta_j} - z^{-n}\,\mathfrak{q}_n^{\eta_j}\big)\Big)\Big(\exp\big(A_{\vec v}(z)_-\big)\, \exp\big(A_{\vec v}(z)_+\big)\Big) \ . \end{aligned} \tag{A.4}$$

For $n > 0$ we arrive at the similar expression

$$\begin{aligned} &\big[L_n^{\widehat{\mathfrak{sl}}_k}, \exp\big(A_{\vec v}(z)_-\big)\, \exp\big(A_{\vec v}(z)_+\big)\big] \\ &\quad = z^n \Big(z\,\partial_z - \frac{1}{2}\,(n-1) \sum_{j=1}^{k-1} v_j^2 + \sum_{j=1}^{k-1} v_j \sum_{t=0}^{n-1} z^{-t}\,\mathfrak{q}_t^{\eta_j}\Big)\Big(\exp\big(A_{\vec v}(z)_-\big)\, \exp\big(A_{\vec v}(z)_+\big)\Big) \\ &\qquad\qquad - \sum_{j=1}^{k-1} v_j\, \exp\big(A_{\vec v}(z)_-\big)\, \Big(\sum_{t=1}^{n} z^{-t}\,\mathfrak{q}_t^{\eta_j}\Big)\, \exp\big(A_{\vec v}(z)_+\big) \ , \end{aligned}$$

but this time the operators $\mathfrak{q}_t^{\eta_j}$ for $t > 0$ do not commute with $\exp\big(A^j_{\vec{v}}(z)_-\big)$. Since

$$\Big[A^j_{\vec{v}}(z)_- \, , \, \sum_{t=1}^{n} z^{-t} \, \mathfrak{q}_t^{\eta_j}\Big] = -n \, v_j \, \mathfrak{c} \, ,$$

we get

$$\Big[\exp\big(A^j_{\vec{v}}(z)_-\big) \, , \, \sum_{t=1}^{n} z^{-t} \, \mathfrak{q}_t^{\eta_j}\Big] = -n \, v_j \, \exp\big(A^j_{\vec{v}}(z)_-\big)$$

and thus we arrive again at Equation (A.4). For $n = 0$ we obtain

$$\big[L_0^{\widehat{\mathfrak{sl}}_k}, \exp\big(A_{\vec{v}}(z)_-\big) \, \exp\big(A_{\vec{v}}(z)_+\big)\big] = z \, \partial_z \Big(\exp\big(A_{\vec{v}}(z)_-\big) \, \exp\big(A_{\vec{v}}(z)_+\big)\Big) \, .$$

Finally, to get the assertion it is sufficient to note that if $D = (d_{ij})$ is the change of basis matrix such that $\gamma_i = \sum_{j=1}^{k-1} d_{ij} \, \eta_j$ with $d_{ij} \in \mathbb{R}$, then $\mathrm{V}^{\gamma_{21}}_{1,-1}(z) = \exp\big(A_{D\vec{v}_{21}}(z)_-\big) \, \exp\big(A_{D\vec{v}_{21}}(z)_+\big)$, and moreover

$$\begin{aligned} \sum_{j=1}^{k-1} \big(D\vec{v}_{21}\big)_j \, \eta_j &= \sum_{i=1}^{k-1} (\vec{v}_{21})_i \, \gamma_i \, , \\ \sum_{j=1}^{k-1} \big(D\vec{v}_{21}\big)_j^2 &= \Big\langle \sum_{j=1}^{k-1} \big(D\vec{v}_{21}\big)_j \, \eta_j \, , \, \sum_{j=1}^{k-1} \big(D\vec{v}_{21}\big)_j \, \eta_j \Big\rangle_{\mathfrak{Q}\otimes_{\mathbb{Z}}\mathbb{R}} = \vec{v}_{21} \cdot C\vec{v}_{21} \, . \end{aligned}$$

□

Remark A.5. A similar (but simpler) calculation shows that the vertex operators $\mathrm{V}_{\alpha,\beta}(z)$ are primary fields in the sense stated in Remark 3.9. △

The proof of Theorem 7.6 is now completed once we establish the following commutation relations.

Lemma A.6. *For any $n \in \mathbb{Z}$ we have*

$$\big[L_n^{\widehat{\mathfrak{sl}}_k}, \bar{\mathrm{V}}_\mu(\vec{v}_{21}, \vec{x}, z)\big] = z^n \, \big(z \, \partial_z + \tfrac{1}{2} \, \vec{v}_{21} \cdot C\vec{v}_{21} \, n\big) \bar{\mathrm{V}}_\mu(\vec{v}_{21}, \vec{x}, z) \, .$$

Proof. To get the assertion, it is enough to derive the commutation relations

$$\begin{aligned} \big[L_n^{\widehat{\mathfrak{sl}}_k}, \exp\big(\log z \, \mathfrak{c} + \gamma_{21}\,\big)\big] = z^n \, \Big(z \, \partial_z + \frac{1}{2} \, \vec{v}_{21} \cdot C\vec{v}_{21} & \\ - \sum_{i=1}^{k-1} \big(\vec{v}_{21}\big)_i \, \big(\mathfrak{q}_0^i - z^{-n} \, \mathfrak{q}_n^i\big)\Big) & \exp\big(\log z \, \mathfrak{c} + \gamma_{21}\,\big) \, , \end{aligned}$$

$$\big[L_0^{\widehat{\mathfrak{sl}}_k}, \exp\big(\log z \, \mathfrak{c} + \gamma_{21}\,\big)\big] = z \, \partial_z \exp\big(\log z \, \mathfrak{c} + \gamma_{21}\,\big) \, .$$

□

B Edge contributions

In this appendix we begin by listing the edge contributions

$$\sum_{n=1}^{k-1} L_{\vec{v}}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)}\big)$$

to the T-equivariant Chern character of the natural bundle $\boldsymbol{V}^{\vec{u},n,j}$ on $\mathcal{M}(\vec{u},n;j)$ which were derived in [14, Appendix C]. For this, we first introduce some notation. Let $j \in \{0,1,\dots,k-1\}$ be the equivalence class of $k\,v_{k-1}$ modulo k. Set $(C^{-1})^{n0}=0$ for $n\in\{1,\dots,k-1\}$ and $(C^{-1})^{k,j}=0$. We also set $\vec{s}:=C^{-1}(\vec{u}-\boldsymbol{e}_j)$ if $j>0$ and $\vec{s}:=\vec{v}$ if $j=0$; then $\vec{s}\in\mathbb{Z}^{k-1}$. We denote by $\lfloor x\rfloor\in\mathbb{Z}$ the integer part and by $\{x\}:=x-\lfloor x\rfloor\in[0,1)$ the fractional part of a rational number x.

If $s_n\geq 0$ for every $n=1,\dots,k-1$, consider the equation

$$\frac{C_{nn}}{2}\,i^2-i\,\Big(\vec{v}-\sum_{p=1}^{n-1}s_p\,\boldsymbol{e}_p\Big)\cdot C\boldsymbol{e}_n+\frac{1}{2}\bigg(\Big(\vec{v}-\sum_{p=1}^{n-1}s_p\,\boldsymbol{e}_p\Big)\cdot C\Big(\vec{v}-\sum_{p=1}^{n-1}s_p\,\boldsymbol{e}_p\Big)-\big(C^{-1}\big)^{cc}\bigg)=0\,, \quad \text{(B.1)}$$

and define the set

$$S_n^+:=\{i\in\mathbb{N}\,|\,i\leq s_n\ \text{ is a solution of Equation (B.1)}\}\,.$$

Let $d_n^+:=\min(S_n^+)$ if $S_n^+\neq\emptyset$ and $d_n^+:=s_n$ otherwise.

When $s_n<0$ for $n=1,\dots,k-1$ consider the equation

$$\frac{C_{nn}}{2}\,i^2+i\,\Big(\vec{v}-\sum_{p=1}^{n-1}s_p\,\boldsymbol{e}_p\Big)\cdot C\boldsymbol{e}_n+\frac{1}{2}\bigg(\Big(\vec{v}-\sum_{p=1}^{n-1}s_p\,\boldsymbol{e}_p\Big)\cdot C\Big(\vec{v}-\sum_{p=1}^{n-1}s_p\,\boldsymbol{e}_p\Big)-\big(C^{-1}\big)^{cc}\bigg)=0\,, \quad \text{(B.2)}$$

and define the set

$$S_n^-:=\{i\in\mathbb{N}\,|\,i\leq -s_n\ \text{ is a solution of Equation (B.2)}\}\,.$$

Let $d_n^-:=\min(S_n^-)$ if $S_l^-\neq\emptyset$ and $d_n^-:=-s_n$ otherwise. Let m be the smallest integer $n\in\{1,\dots,k-1\}$ such that S_n^+ or S_n^- is nonempty; if all of these sets are empty, let $m:=k-1$.

Then for fixed $n=1,\dots,m$ we set:

- For $v_n-(C^{-1})^{nj}>0$:
  - For $\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}+2(v_n-(C^{-1})^{nj}-d_n^+)\geq 0$:

$$L_{\vec{v}}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)}\big)=-\sum_{i=v_n-(C^{-1})^{nj}-d_n^+}^{v_n-(C^{-1})^{nj}-1}\ \sum_{j=0}^{2i+\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}}\big(\chi_1^n\big)^{i+\left\lfloor\frac{\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}}{2}\right\rfloor}\big(\chi_2^n\big)^j\,.$$

  - For $2\leq\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}+2(v_n-(C^{-1})^{nj})<2d_n^+$:

$$L_{\vec{v}}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)}\big)$$

$$= \sum_{i=v_n-(C^{-1})^{nj}-d_n^+}^{-\left\lfloor \frac{\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}}{2} \right\rfloor -1} \sum_{j=1}^{2i-(\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j})-1} \left(\chi_1^n\right)^{i-\left\lfloor -\frac{\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}}{2} \right\rfloor} \left(\chi_2^n\right)^{-j}$$

$$- \sum_{i=-\left\lfloor \frac{\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}}{2} \right\rfloor}^{2(v_n-(C^{-1})^{nj})+\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}-2} \sum_{j=0}^{2i+\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}} \left(\chi_1^n\right)^{i+\left\lfloor \frac{\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}}{2} \right\rfloor} \left(\chi_2^n\right)^{j}.$$

- For $\delta_{n,j} - v_{n+1} + (C^{-1})^{n+1,j} < 2 - 2(v_n - (C^{-1})^{nj})$:

$$L_{\vec{v}}^{(n)}\left(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}\right) = \sum_{i=v_n-(C^{-1})^{nj}-d_n^+}^{v_n-(C^{-1})^{nj}-1} \sum_{j=1}^{-2i-\delta_{n,j}+v_{n+1}-(C^{-1})^{n+1,j}-1} \left(\chi_1^n\right)^{i-\left\lfloor -\frac{\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}}{2} \right\rfloor} \left(\chi_2^n\right)^{-j}.$$

- For $v_n - (C^{-1})^{nj} = 0$: $\quad L_{\vec{v}}^{(n)}\left(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}\right) = 0$.

- For $v_n - (C^{-1})^{nj} < 0$:

  - For $\delta_{n,j} - v_{n+1} + (C^{-1})^{n+1,j} + 2v_n - 2(C^{-1})^{nj} < 2 - 2d_n^-$:

$$L_{\vec{v}}^{(n)}\left(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}\right) = - \sum_{i=1-v_n+(C^{-1})^{nj}-d_n^-}^{-v_n+(C^{-1})^{nj}} \sum_{j=1}^{2i-(\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j})-1} \left(\chi_1^n\right)^{-i-\left\lfloor -\frac{\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}}{2} \right\rfloor} \left(\chi_2^n\right)^{-j}.$$

  - For $2 - 2d_n^- \leq \delta_{n,j} - v_{n+1} + (C^{-1})^{n+1,j} + 2v_n - 2(C^{-1})^{nj} < 0$:

$$L_{\vec{v}}^{(n)}\left(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}\right) = \sum_{i=1-v_n+(C^{-1})^{nj}-d_n^-}^{\left\lfloor \frac{\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}}{2} \right\rfloor} \sum_{j=0}^{-2i+\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}} \left(\chi_1^n\right)^{-i+\left\lfloor \frac{\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}}{2} \right\rfloor} \left(\chi_2^n\right)^{j}$$

$$- \sum_{i=\left\lfloor \frac{\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}}{2} \right\rfloor+1}^{-v_n+(C^{-1})^{nj}} \sum_{j=1}^{2i-(\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j})-1} \left(\chi_1^n\right)^{-i-\left\lfloor -\frac{\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}}{2} \right\rfloor} \left(\chi_2^n\right)^{-j}.$$

  - For $\delta_{n,j} - v_{n+1} + (C^{-1})^{n+1,j} \geq -2v_n + 2(C^{-1})^{nj}$:

$$L_{\vec{v}}^{(n)}\left(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}\right) = \sum_{i=1-v_n+(C^{-1})^{nj}-d_n^-}^{-v_n+(C^{-1})^{nj}} \sum_{j=0}^{-2i+\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}} \left(\chi_1^n\right)^{-i+\left\lfloor \frac{\delta_{n,j}-v_{n+1}+(C^{-1})^{n+1,j}}{2} \right\rfloor} \left(\chi_2^n\right)^{j}.$$

For $n = m+1, \dots, k-1$ we set $L^{(n)}_{\vec{v}}\big(\varepsilon^{(n)}_1, \varepsilon^{(n)}_2\big) = 0$. Note that for any fixed $n \in \{1, \dots, k-1\}$, $d^{\pm}_n = 0$ implies $L^{(n)}_{\vec{v}}\big(\varepsilon^{(n)}_1, \varepsilon^{(n)}_2\big) = 0$.

The edge factors $\ell^{(n)}_{\vec{v}_{21}}\big(\varepsilon^{(n)}_1, \varepsilon^{(n)}_2, \mu\big)$ which contribute to the T-equivariant Euler class of the Carlsson-Okounkov bundle $\boldsymbol{E}^{\vec{u}_1,n_1,j_1;\vec{u}_2,n_2,j_2}_{\mu}$ on $\mathcal{M}(\vec{u}_1, n_1; j_1) \times \mathcal{M}(\vec{u}_2, n_2; j_2)$ are then obtained in the following way. We replace $\vec{v}$ in the above by $\vec{v}_{21}$ and j by j_{21}. If

$$L^{(n)}_{\vec{v}_{21}}\big(\varepsilon^{(n)}_1, \varepsilon^{(n)}_2\big) = \sum_{i=1}^{D} \eta_i \, \mathrm{e}^{\,\sigma_i}$$

with $\eta_i = 0, \pm 1$, then

$$\ell^{(n)}_{\vec{v}_{21}}\big(\varepsilon^{(n)}_1, \varepsilon^{(n)}_2, \mu\big) = \prod_{i=1}^{D} (\mu + \sigma_i)^{\eta_i} \ .$$

Explicit formulas are written in [14, Section 4.7].

The contribution of $L^{(n)}_{\vec{v}_{21}}\big(\varepsilon^{(n)}_1, \varepsilon^{(n)}_2\big)$ to the p-th equivariant Chern class $(\mathrm{c}_p)_T\big(\boldsymbol{V}^j\big)$ is gotten by extracting the monomial terms of total degree p in $\varepsilon_1, \varepsilon_2$. In particular, the contribution to the first Chern class is given by

$$\ell^{(n)}_{\vec{v}}\big(\varepsilon^{(n)}_1, \varepsilon^{(n)}_2\big)_{[1]} = \Big(\varepsilon_1 \ \frac{\partial}{\partial \varepsilon_1}\Big|_{\varepsilon_1=0} + \varepsilon_2 \ \frac{\partial}{\partial \varepsilon_2}\Big|_{\varepsilon_2=0}\Big) L^{(n)}_{\vec{v}_{21}}\big(\varepsilon^{(n)}_1, \varepsilon^{(n)}_2\big) \ .$$

Example B.3. Let $k = 2$. Then $j_{21} \in \{0, 1\}$, while $\{v_{21}\} = \frac{1}{2}\,\delta_{1,j_{21}}$ and $\lfloor v_{21} \rfloor = v_{21} - (C^{-1})^{1,j_{21}}$. Since $m = 1$, $d^+_1 = \lfloor v_{21} \rfloor$ and $d^-_1 = -\lfloor v_{21} \rfloor$, and we get

$$\ell_{v_{21}}(\varepsilon_1, \varepsilon_2, \mu) = \begin{cases} \displaystyle\prod_{i=0}^{\lfloor v_{21} \rfloor - 1} \ \prod_{j=0}^{2i+2\{v_{21}\}} \big(\mu + i\,\varepsilon_1 + j\,\varepsilon_2\big) & \text{for } \lfloor v_{21} \rfloor > 0 \ , \\ 1 & \text{for } \lfloor v_{21} \rfloor = 0 \ , \\ \displaystyle\prod_{i=1}^{-\lfloor v_{21} \rfloor} \ \prod_{j=1}^{2i-2\{v_{21}\}-1} \big(\mu + (2\{v_{21}\} - i)\,\varepsilon_1 - j\,\varepsilon_2\big) & \text{for } \lfloor v_{21} \rfloor < 0 \ . \end{cases}$$

For $\{v_{21}\} = 0$ these formulas coincide with the blowup factors obtained in [28] up to a redefinition of the equivariant parameters (see also [20]). Moreover, for $\lfloor v_{21} \rfloor > 0$ they can be easily written in the form

$$\ell_{v_{21}}(\varepsilon_1, \varepsilon_2, \mu) = \prod_{\substack{i,j \geq 1\,,\, i+j \leq 2\lfloor v_{21} \rfloor \\ i+j \equiv 0 \bmod 2}} \big(\mu + (i-1)\,\tilde{\varepsilon}_1 + (j-1)\,\tilde{\varepsilon}_2\big)$$

with $\tilde{\varepsilon}_1 = \frac{\varepsilon_1}{2}$ and $\tilde{\varepsilon}_2 = \frac{\varepsilon_1}{2} + \varepsilon_2$, which coincide with the blowup factors of [10, 11, 7] (similarly for $\lfloor v_{21} \rfloor < 0$ and/or $\{v_{21}\} = \frac{1}{2}$). In [7] it is stated that these edge factors can be represented as suitable matrix elements of primary fields from Theorem 7.6 in highest weight states of $\widehat{\mathfrak{gl}}_2 = \mathfrak{h} \oplus \widehat{\mathfrak{sl}}_2$ at level one; the proof makes use of the Frenkel-Kac construction and the Dotsenko-Fateev integrals of [24].

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